Low-energy Lorentz symmetry violation from quantum corrections in Lifshitz-scaling models

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A thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy in Physics

3^{rd} of January, 2015

Abstract

In this thesis we investigate the effects of low-order quantum corrections on Lifshitz-type quantum field theories. In particular, we consider the Lorentz-symmetry violating corrections to the dispersion relations of the various particles of these theories at low energies, which may be of a significant size even where the classical effect is small.

We first study a Lifshitz scaling model of two fermion flavours in flat space, interacting by a flavour mixing four-point term. We demonstrate the dynamical generation of masses and flavour oscillations and consider these as a possible model of neutrino mixing. We use existing experimental constraints on neutrino masses and mixing angles to place restrictions on the couplings of our model. We then investigate quantum corrections to the couplings and dispersions, and find that the latter would be too large to be considered physical.

Next, we investigate a Lifshitz scaling model of Quantum Electrodynamics, containing only fermions and gauge fields. We investigate the extent to which such models may be phenomenologically viable, again primarily through calculating low-order dressed dispersion relations. In doing this, we encounter issues not seen in the simpler model such as those of gauge fixing and dimensional regularisation in anisotropic theories. We again find that the dressed dispersion relations appear notably non-relativistic even at low energies, despite the classical model being well within experimental bounds.

Finally, we investigate the dressing of scalar and vector boson dispersion relations by the quantum effects of the so-called "covariant" extension of Hořava-Lifshitz gravity, which despite having an unusual extra symmetry seems better behaved than the "original" form of Hořava gravity. We find that even integrating out the effects of quantum gravity fluctuations alone gives significant corrections to the matter sector's dispersion relations, which allows us to place some new constraints on the energy scales of the theory.

Declaration

I confirm that the work presented here is my own and that all references are cited accordingly.

Chapter 5 contains work done in collaboration with Jean Alexandre and Nicholas Houston and published in [1].

Chapters 6 and 8 contain work done in collaboration with Jean Alexandre and published in [2] and [3].

Acknowledgements

I would like to thank my supervisor Jean Alexandre for his patience these past years. I would also like to thank my fellow PhD students in the physics department at King's College, for putting up with my inane questions for so long; especially Nicholas Houston, who bore the brunt of it.

Contents

Chapter 1

Introduction

1.1 Motivations

A severe restriction often imposed on a quantum field theory (QFT) is the requirement, if the theory is to be treated as anything other than an effective approximation, that it be renormalisable into the far ultraviolet; in practice, we usually require our models to be *perturbatively* renormalisable, as this is generally our only practical way of calculating meaningful results.

The majority of observed phenomena can be accounted for by the Standard Model of particle physics, which is believed to be renormalisable, plus the theory of General Relativity (GR). However, GR, treated as a perturbative quantum field theory about some background metric, fails to be perturbatively renormalisable and so (for this reason, and complications arising from curved spacetime) cannot be readily treated as a QFT. Barring the possibility of some non-gaussian fixed point, to which the theory may flow in the ultra-violet without it being detectable perturbatively (the "Asymptotic Safety" scenario), one therefore expects the present models of gravity to break-down in some way above a certain energy scale, perhaps requiring something "beyond" standard relativistic QFT.

Lorentz symmetry, global or local, is a well-founded and well-tested characteristic of all our current predictive models of fundamental physics, over a wide range of scales. However, it is a relatively simple thing to break, either deliberately or through some unexpected condensation; indeed, many speculative models of more fundamental physics involve some degree of Lorentz symmetry violation, usually appearing at very high energies so as not to produce effects at experimental energies that should already have been observed.

A type of Lorentz violating model of that has garnered interest lately, and is the primary focus of this study, is those with so-called Lifshitz scaling. In these models there is a preferred choice of time co-ordinate, which has differing mass dimension and asymptotic scaling to the spatial co-ordinates. These models are of interest as they allow many more interactions in quantum field theories to become renormalisable. In particular, a recent proposal by Hořava [4] for a Lifshitz-scaling model of gravity, appears to be perturbatively renormalisable, at least by power-counting; such a model, if it could be made to approximate GR at experimental energies (a non-trivial condition), would allow us to treat gravity with our usual methods of perturbative QFT.

Throughout this thesis, we shall see that it is not as easy to sequester Lorentz-violating effects away at high energies as is often assumed. We shall investigate various models with Lifshitz scaling (and several of the usually non-renormalisable interactions that would not be allowed in a Lorentz invariant model) and calculate low-order quantum corrections to some important processes, we shall focus especially on the anisotropic corrections to the propagation relations of the various species of particle in the models as these may lead to readily observable deviations from relativistic behaviour. We shall find repeatedly that quantum considerations give much stronger constraints on the scales of Lorentz violation than a purely classical analysis would suggest.

1.2 Structure

We begin by introducing the theoretical concepts behind Lorentz violating models and some of the techniques used to study these. We discuss Lifshitz-scaling models in flat spacetime. We then study a model of fermions, taken to represent neutrinos, with an interaction that would not be allowed in an isotropic theory. We demonstrate the dynamical generation of masses and flavour oscillations in this model and calculate lowest-order quantum corrections to the propagation relations. We find that the induced low-energy Lorentz violation will likely be too large to be realistic, without significant fine-tuning.

We then study similar effects in a more complex model: a Lifshitz-scaling version of quantum electrodynamics, with one fermion coupled to a $U(1)$ gauge field. We see that the simultaneous imposition of Lifshitz scaling and gauge symmetry introduces new interactions and complicates gauge fixing. We examine some unusual features of the method of dimensional regularisation (that we adopt here as it preserves the gauge symmetry) that one would not see in isotropic QED.

At this point we discuss Lifshitz scaling in curved spacetime and introduce Hořava-Lifshitz gravity (and its various extensions and modifications). We then perform a similar analysis to the first two studies in the "covariant" extension of Hořava-Lifshitz gravity, in which an extra $U(1)$ symmetry is introduced to restore the number of propagating degrees-of-freedom per spacetime point to its relativistic value. We study this model coupled to "normal" classical Lorentz-invariant matter and integrate out quantum metric fluctuations about flat space to produce an effective action for the matter. We again find a significant Lorentz-violating effect at low energies which can be used to further restrict the range of possible parameters of such models; in particular the scale of Lorentz violation is given bounds that are physically reasonable.

The structure of this thesis is as follows:

- In chapter 2 we introduce the basic concepts behind Lorentz violation in general, and discuss various models that exhibit such behaviour.
- In chapter 3 we discuss the effective action methods that we shall use repeatedly in subsequent chapters
- In chapter 4 we consider Lifshitz models in particular and examine their unusual features, including the mechanisms by which they improve the renormalisability of QFTs.
- In chapter 5 we examine a Lifshitz model for two species of fermion, with a flavour-mixing 4-point interaction that would be nonrenormalisable in a relativistic theory. We calculate one-loop corrections to the propagation speed and examine the dynamical generation of masses and flavour oscillations.
- In chapter 6 we perform a similar analysis for a more realistic abelian gauge theory, containing both fermions and gauge bosons. We find corrections to apparent low-energy propagation speeds, with some interesting mathematical features, which may impose restrictions on the allowed parameters of the model.
- In chapter 7 we discuss the violation of local Poincaré symmetry in the context of Hořava's Lifshitz-scaling model of gravity; its assumptions, experimental bounds, the several versions that have been more recently proposed and their various issues. We also relate Hořava's model to other modified gravity scenarios.
- In chapter 8, we consider so-called "covariant" Hořava-Lifshitz gravity coupled to relativistic matter and integrate out the effects of gravitational fluctuations about flat space; again we obtain modified propagation relations that allow

us to restrict several parameters of the model by comparison to experimental data.

- Chapter 9 contains a summary of the work, concluding remarks and potential directions for further studies.
- This thesis includes three appendices; the first two detail longer calculations omitted from chapters 5 and 6. The final appendix discusses some unusual features of the method of dimensional regularisation.

1.3 Notation and conventions

If not specified otherwise, we shall assume that we are working in $3+1$ spacetime dimensions.

As we shall be discussing the violation of Lorentz symmetry, we shall need to distinguish space and time co-ordinates: we use lower-case Greek indices, $\mu, \nu \ldots$, for the components of spacetime tensors, lower-case Latin indices, $i, j, \ldots$ for spatial tensors (or spatial components of spacetime tensors) and 0 for time components. We adopt the usual summation conventions over repeated indices, for both space and spacetime tensors.

We take $\Delta = \partial^i \partial_i$ to be the *spatial* Laplacian. Similarly for a 4-vector $k_\mu = (k_0, k_i)$ we write k as the magnitude of its *spatial part*. We will generally adopt the convention that $c = 1 = \hbar$ and use the mostly-minus metric signature (such that $k_\mu k^\mu = k_0^2 - k^2$). We shall occasionally write a spatial vector as $\vec{k}$, where we feel it may be confused with a scalar.

Curved spacetime

In discussions of Hořava-Lifshitz gravity, we shall adopt the mostly-plus convention more commonly used by others in the field of quantum gravity (and general relativity in general), in order to avoid confusion when comparing our work to that of other authors. We shall, of course, make very clear when this change of convention is happening.

In curved spacetime, we shall need to distinguish spatial tensors contracted with the ADM 3-metric g_{ij} from those contracted with the flat-space 3-metric δ_{ij}. For two vectors v_i, w_i, we will write the former as $v^i w_i$ and the latter as $v_i w_i$, which leads to $v^\mu w_\mu = -v_0 w_0 + v^i w_i$, where

$$\begin{aligned} v^i w_i &= v_i w_j g^{ij} \ , \\ g^{ij} g_{jk} &= \delta_{ik} \ . \end{aligned} \tag{1.1}$$

Also, we denote $v^2 = v_i v_i$, we use ∂_i for the flat 3-space derivative, and ∂^2 for the flat 3-space Laplacian.

Chapter 2

Background on Lorentz violation

In this chapter we introduce the basic features of Lorentz symmetry and the various effects of its violation, such as the non-relativistic propagation that will be the main focus of this thesis. Finally, we examine several models that include Lorentz violation, including the "Standard Model Extension" that can be taken as a parametrisation of most Lorentz-violating terms.

2.1 Lorentz and Poincaré symmetry

The Lorentz group is the maximal set of linear transformations that preserve the Minkowski metric $\eta_{\mu\nu} = diag\{1, -1, -1, -1\}$ and the origin; its generators are antisymmetric matrices $M_{\mu\nu}$ obeying

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(M_{\mu\rho}\eta_{\nu\sigma} + M_{\nu\sigma}\eta_{\mu\rho} - M_{\mu\sigma}\eta_{\nu\rho} - M_{\nu\rho}\eta_{\mu\sigma}) \ . \tag{2.1}$$

Equivalently, the (proper, orthochronous) Lorentz group is the unique set of symmetries such that a linear transformation relates different inertial frames in a reversible and isotropic fashion, depending only on the velocity, while preserving a constant speed (that of massless particles). The Galilean transformations of Newtonian physics can be considered the $c \to \infty$ limit of the Lorentz transformations. With the addition of the generators of space-time translations, P_μ, these form the Poincaré algebra:

$$[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu). \tag{2.2}$$

We shall follow most of the literature in referring primarily to "Lorentz symmetry" and "Lorentz violation", rather than Poincaré symmetry; the non-invariance of physical laws under translation is not often considered. This distinction becomes slightly more important when considering *local* Poincaré symmetry and its possible violations.

2.2 Lorentz symmetry breaking

Conceptually, the violation of Lorentz symmetry is very easy. One can simply add some coupling to a constant non-Lorentz-invariant background field, such as a constant vector or spinor, to an existing model (or one can introduce a potential that will generate the same spontaneously). Generally, when "Lorentz violation" is mentioned, what is meant is a violation of invariance under the *boost* transformations (i.e. the generators M_{0i}) in some frame; this could be justified by noting that the natural parameter for boosts is the relative velocity which, unlike the angle of rotation associated with the M_{ij}, cannot be so readily tested throughout its whole range or, relatedly, that large boosts for massive objects necessarily contain a high energy scale, which could lead to effects unobserved at typical experimental energies.

Generally, while breaking boosts produces an asymmetry between space and time, it does *not* restore the Galilean symmetry of Newtonian dynamics. Without additional assumptions, the Poincaré group will simply be broken into the $SO(3)$ of spatial rotations, plus space and time translations. This sort of Lorentz breaking suggests there is a *preferred rest frame* in which "space" and "time" co-ordinates may be separated [1]; in most such models, this may be taken to be the rest frame of the Cosmic Microwave Background (CMB), unless some frame-dragging effect is invoked. However, most experimental searches for Lorentz violation adopt a more convenient Sun-centred frame.

There are several other generic properties of theories without boost symmetry. One is that if the Lorentz breaking terms are coupled to one species of particle but not another (or with different strengths), then propagation relations may differ between those species (in ways other than the usual differing masses), such as differing effective "speeds of light" which will be discussed in more detail below. Alternatively, the propagation relation of even a single species may become energy-dependent in a way other than the expected

$$\omega^2 = m^2 + k^2 \,, \tag{2.3}$$

where the frequency $\omega = k_0$. In fact, energy-dependent propagation relations may allow one to construct models in which propagation is approximately relativistic at low energies (where we have measured it) while being highly Lorentz-violating at higher scales. Such a model would come at the cost of introducing at least one

[1] For instance, if there were a time- or space-like background vector, the frame in which it has no space or time components (respectively) would seem to be a natural rest frame. It should be noted that such a vector is not sufficient to define a global foliation of a spacetime into space-like surfaces. A null background vector may well violate Lorentz invariance, but not give a preferred rest frame.

typical *energy scale* to the model. For example, a particle with the propagation relation

$$\omega^2 = m^2 + k^2 + \frac{k^4}{M^2} \tag{2.4}$$

would appear Lorentz invariant when $k \ll M$, but not when $k \gg M$. Such relations are common in the Lifshitz models we shall discuss later.

A distinction should be made between Lorentz violation in classical and quantum field theories. In the former, it is relatively simple to confine Lorentz-violating effects to a few (weakly-interacting) species or very high energies but once quantum effects are taken into account this becomes significantly more difficult: loop corrections involve integrating over arbitrarily high loop momenta, so "high energy" effects from non-Lorentz-invariant propagators may appear even at low energies (suppressed by powers of $\hbar$, but that may not be sufficient to render them unobservable at experimental energies).

Of particular relevance to the present study are quantum corrections to low-energy dispersion relations. Without Lorentz invariance, the time- and space-derivative terms in the kinetic part of the action for some particle may be dressed differently, leading to a dispersion relation of the form

$$A\omega^2 = m^2 + Bk^2 \ . \tag{2.5}$$

In relativistic QFT, one would have $A = B$, and so this correction could be absorbed by a re-definition of the field strengths (this is the usual "wavefunction renormalisation" of QFT); clearly, that is not possible here. Of course, one could always perform a re-scaling of space or time to restore the usual relativistic dispersion relation (while modifying the mass). However, if one had two or more species that are dressed differently, performing such a rescaling for both species simultaneously would be impossible. Such a scenario leads to the two species having dispersion relations

$$\begin{aligned} \omega_1^2 &= m_1^2 + k_1^2, \\ \omega_2^2 &= m_2^2 + Ak_2^2. \end{aligned} \tag{2.6}$$

The two species have different effective "speeds of light" (1 and $\sqrt{A}$ respectively)[2] and associated "light cones" [5]. As these are quantum corrections, they may run with energy, leading additionally to an energy dependence in the propagation.

It should be noted that without the full Poincaré symmetry, the energy-momentum tensor is not a conserved quantity, though any of its components may be, separately,

[2]Though the choice of which species' speed to set to 1 is, of course, arbitrary.

depending on the remaining symmetries.

It should also be noted that any violation of CPT invariance would also imply Lorentz violation [6].

2.3 Lorentz-violating models

2.3.1 The Standard Model Extension

In [7], the authors have collected experimental restrictions on the coefficients of the so-called Standard Model Extension (SME), an effective theory consisting of the Standard Model of particle physics, general relativity and all possible relevant Lorentz- or Poincaré-violating operators, for both the Standard Model fields and gravity [8]; the coefficients of these operators therefore provide an effective parametrisation of possible Lorentz-violating effects.
For instance, if we restrict our attention to a single fermion, ψ, in flat spacetime, a minimal model that includes dimension 3 and 4 operators would give a Lagrangian density of

$$\begin{aligned} \mathcal{L}_{LV} &= \overline{\psi}\big(A_{\mu\nu}\sigma^{\mu\nu} \\ &+ a_\mu\gamma^\mu + b_\mu\gamma_5\gamma^\mu \\ &+ c_{\mu\nu}\gamma^\mu D^\nu + d_{\mu\nu}\gamma_5\gamma^\mu D^\nu \\ &+ e_\mu D^\mu + f_\mu\gamma_5 D^\mu + g_{\mu\nu\xi}\sigma^{\mu\nu}D^\xi\big)\psi \ , \end{aligned} \tag{2.7}$$

in addition to the usual kinetic and mass terms. The coefficients A, a, b, c, d, e, f, g are arbitrary tensors that may not be Lorentz invariant (indeed several of them, by symmetry, cannot be unless they are zero) and D_μ is the covariant derivative under whatever gauge symmetries the model may possess. Note that only the terms on the second and fourth rows above violate CPT invariance.

Even the simple model above contains 78 scalar parameters, so it should be clear that the more general SME can contain a very large number of terms, especially once higher-dimension operators are allowed. The most readily measured dimensionless coefficients can be seen to be of order 10^{-15} at most, so any Lorentz violating effect must either be negligible at current experimental energies or affect only hard-to-measure processes.

The data from which these limits are derived come from a wide variety of sources including

- astrophysical data such as time-of flight measurements from supernovae and quasars
- precision measurements of cold atoms and optical resonators
- vacuum birefringence measurements from distant light sources
- Earth-based experiments searching for annual variations in physical processes
- direct tests for CPT violation and other collider physics.

The references in [7] provide a fairly comprehensive list of experiments that we shall not repeat here. Instead we shall examine one particular example in some detail below, to illustrate the typical concepts.

Example test of SME parameters - electrodynamics

In [9], the authors examine a Lorentz-violating extension to the gauge sector of electrodynamics, with the aim of designing an experiment to improve the existing bounds on such models, their model is parametrised by the 23 independent parameters of a and b in

$$\mathcal{L}_{LV} = a_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + b^{\lambda}\epsilon_{\lambda\mu\nu\rho}A^{\mu}F^{\nu\rho} \ , \tag{2.8}$$

and they find that *all but three* of these parameters can be restricted to be less than 10^{-37} simply by the non-observation of birefringence effects from distant astrophysical sources; in fact the components of b must be less than 10^{-42} GeV. The three remaining three parameters are the components of

$$v_i = \frac{1}{2}(a_{0kik} - a_{0kki}) \tag{2.9}$$

which are "only" constrained to be less than 10^{-11}, from optical cavity experiments. Setting the other, more restricted terms to zero, the authors then calculate the alterations to Maxwell's equations: they find that a non-zero v_i will produce a magnetic field from a static charge, and an electric field from a steady current. Having determined this, the authors conclude that such an effect would be easily observed in charged particle beam interference experiments; they describe a simple beam splitting and re-combination experiment in which the presence of v_i induces a path-dependent phase shift that is, in principle, measurable and should allow one to improve the bound on the v_i by one or two orders of magnitude, given reasonable experimental accuracies.

It should be noted that this experiment measures Lorentz violation in the QED sector, where particles are comparatively strongly interacting and easily observed.

Constraints for more weakly coupled fields, such a neutrinos and the gravity sector, tend to be looser. In chapter 7, we shall examine results from such an effective model, derived from Hořava-Lifshitz gravity.

2.3.2 "Top-down" models with Lorentz violation

The SME discussed above is only an effective phenomenological model. We will now briefly describe two more "fundamental" Lorentz-violating models, motivated by theoretical concerns, that may be related to the Lifshitz theories that are the main focus of this study.

Any model that attempts to discretise space and time will violate Lorentz invariance: there is no possible Lorentz invariant regular lattice, nor (under certain reasonable assumptions) can there be a Lorentz invariant random distribution of points [10]. In particular, the approach of *Causal Dynamical Triangulations* is to approximate continuous spacetime with a discrete network of simplices, such that all the "curvature" is contained in the deficit angles between triangles [11][12]. In [12] a phase of spacetime in Causal Dynamical Triangulations is found that appears to be singular unless one imposes Lifshitz-like scaling in the limit approaching it.

Doubly Special Relativity [13] refers to a collection of models that preserve a second fundamental scale, normally taken to be the Planck energy. Generically, these theories lead to energy-dependent dispersion relations. *Doubly General Relativity* or *Rainbow Gravity* [14] is an extension of these ideas to curved spacetime (in [15] a correspondence between Doubly General Relativity and Hořava gravity is established).

Other models, such as *Einstein-Aether gravity*, are best examined as theories in curved spacetime, and so we shall delay their discussion until we have introduced Hořava-Lifshitz gravity in chapter 7.

Chapter 3

Effective actions and kinetic term dressing in Lorentz-violating theories

In this chapter, we introduce effective actions and describe a few of the mathematical notions and techniques that will be of use in later chapters.

3.1 Effective actions

When we discuss an "effective action" here we shall usually mean it in the sense of an action for a classical field that gives the expectation of the equivalent quantum field. Given a generating functional, $Z[J]$ for a QFT, which we shall take to have just one scalar field, $\phi(x,t)$ for simplicity (and thus only one source term $J(x,t)$)

$$Z[J] = \int D\phi e^{i(S(\phi)+\int d^3xdtJ\phi)} \ . \tag{3.1}$$

We define $W[J] = i \log Z[J]$ which is, by simple analysis of the properties of the exponential, the generating functional of *connected* correlation functions. The effective action we define by

$$\Gamma[\phi_c] = -W[J] - \int d^3xdtJ\phi_c \tag{3.2}$$

where $\phi_c(x,t) = -\frac{\delta W[J]}{\delta J(x,t)}$ is the expectation of ϕ and can be considered the classical field.

3.1.1 Calculating effective actions

Of course, calculating Γ exactly would be equivalent to solving the theory [1], which is not tractable in most cases. We instead approximate the action to one or

[1] as one would "simply" have an extremely non-linear classical field theory in which to calculate

two loop order.

Let the action S, be an integral of some classical Lagrangian density (that is, without renormalisation counterterms), $\mathcal{L}$. We attempt to calculate the effective action perturbatively; we write $\phi = \phi_c + \varphi$, where φ represents fluctuations about the classical solution. We can now represent $Z[J]$ by

$$\int D\varphi \exp(i \int d^3xdt[\mathcal{L}(\phi_c) + J\phi_c + (0)\varphi + \int d^3x'dt' \frac{1}{2}\varphi\varphi' \frac{\delta^2 \mathcal{L}}{\delta\phi\delta\phi'} + \ldots]) \tag{3.3}$$

where ϕ', φ' are functions of x', t', the terms linear in φ vanish by the definition of ϕ_c and all functional derivatives are evaluated at $\phi_c(x)$. The ellipsis represents both higher order terms in φ and possible renormalisation counterterms.

If we truncate this expansion at order φ^2, we may perform the path integral exactly (as it is a Gaussian integral). We obtain

$$Z[J] \simeq \exp(i \int d^3xdt[\mathcal{L}(\phi_c) + J\phi_c).(\det[-\frac{\delta^2 \mathcal{L}}{\delta\phi(x)\delta\phi(x')}])^{-1/2}, \tag{3.4}$$

where a functional determinant can be suitably defined as

$$\begin{aligned} \det f(x,y) &= \exp \operatorname{Tr} \log f(x,y) \\ \operatorname{Tr}[f(x,y)] &= \int d^dxd^dy f(x,y)\delta(x-y) \end{aligned} \tag{3.5}$$

Once we have evaluated this determinant, and added any necessary counterterms [2], $\delta\mathcal{L}$, our first order approximation to the effective action can be written as

$$\Gamma_1[\phi_c] = \int d^3xdt[\mathcal{L} + \delta\mathcal{L}](\phi_c) + \frac{i}{2} \log\det[-\frac{\delta^2\mathcal{L}}{\delta\phi(x)\delta\phi(x')}_{\phi=\phi_c}]. \tag{3.6}$$

Iterating this process would produce higher-order corrections. In practice, calculating even this approximation to the full effective action is often not necessary, as the information one seeks can be found by making simplifying assumptions (such as constant ϕ_c to find the effective potential for the ground state).

3.1.2 Γ as a generating functional and diagrammatics

Given that

$$\frac{\delta^2 W[J]}{\delta J(x)\delta J(x')} = -iD(x,x') \tag{3.7}$$

[2]in particular, to remove tadpole terms, which would invalidate our assumption that φ is perturbative

gives the propagator, by the chain rule and the definition of ϕ_c we have

$$\frac{\delta}{\delta J(x)} = \int dy\ iD(x,y)\frac{\delta}{\delta\phi_c(y)} \tag{3.8}$$

and also

$$\frac{\delta^2\Gamma[\phi_c]}{\delta\phi_c(x)\delta\phi_c(x')} = iD^{-1}(x,x')\ . \tag{3.9}$$

Considering a third derivative of W (i.e a connected 3-point function), one obtains

$$\frac{\delta^3 W[J]}{\delta J(x)\delta J(y)\delta J(z)} = \int dudvdw\ iD(x,u)D(v,y)D(w,z)\frac{\delta^3\Gamma[\phi_c]}{\delta\phi_c(u)\delta\phi_c(v)\delta\phi_c(w)} \tag{3.10}$$

that is to say, the third functional derivative of Γ gives the connected three-point function with the external propagators removed i.e the one-particle irreducible three point function. One can iterate this calculation for higher derivatives of W, where one can see by simple induction that $\Gamma[\phi_c]$ is indeed *the generating functional of one-particle-irreducible correlation functions.* This is extremely useful, as it allows one to rapidly determine the relevant terms for an n-loop approximation to the effective action, simply from 1PI Feynman diagrams.

3.2 Calculating the dispersion relation corrections

We can now expand

$$\Gamma[\phi_c] = \int d^3x dt \left[V_{eff}(\phi_c) + Z_{\mu\nu}(\phi_c)\partial_\mu\phi_c\partial_\nu\phi_c + O(\partial^4)\right] \tag{3.11}$$

V_{eff} is the usual effective potential.

The second term above, $Z_{\mu\nu}$ will be of most interest to us in this study, as it contains the terms that will produce the effective low-energy dispersion relations for the quantised theory. In a Lorentz invariant theory, we would know that $Z_{\mu\nu} = Z\eta_{\mu\nu}$ for some scalar function Z, (the lowest order in ϕ_c of which is the usual wavefunction renormalisation). Without Lorentz invariance there is no such guarantee, and this may lead to modified dispersion relations as discussed in chapter 2.

Conceptually, calculating the lowest-order terms in $Z_{\mu\nu}$ is no different to the usual method of calculating the effective potential; indeed, one should consider the same 1PI graphs, but instead of assuming the external momenta vanish, take them at some finite value k_μ (for the lowest order terms in ϕ_c, which are the ones that produce corrections to the low-energy dispersion relations, we need only consider two-legged graphs and thus only need one external momentum). The relevant terms are then those proportional to k^2 in a Taylor expansion. We can see this easily by re-writing the effective action in terms of Fourier components $\tilde\phi(k)$:

$$\Gamma = \int d^3k dk_0 \left[\tilde V + \tilde Z_{\mu\nu}k^\mu k^\nu\tilde\phi^2 + O(k^4,\tilde\phi^3)\right]\ , \tag{3.12}$$

where we have implicitly expanded in both k and $\tilde{\phi}$ to obtain a constant $\tilde{Z}_{\mu\nu}$.

We therefore need to expand the relevant terms both to the relevant order in the couplings (or $\hbar$) and to second order in the external momentum.

Of course, for fermions, it would be the terms of *linear* order in the external momentum that were of interest to us.

See section 4.6 for a worked example of these methods.

3.2.1 Effect on superficial divergences

As we are expanding to second order in the external momentum, k, we should expect the relevant graphs to have a degree of divergence two less than would be implied by naïve momentum power counting. Consider, for instance, the following integral that may appear in loop corrections in a Lorentz invariant theory

$$\begin{aligned} &\int d^4r \tfrac{r^n}{(r+k)^2+m^2} \\ &= \int d^4r r^n \left(\tfrac{1}{r^2+m^2} - \tfrac{k^2}{(r^2+m^2)^2} + \tfrac{4(r_\mu k^\mu)^2}{(r^2+m^2)^2} \right) + O(k^3) \end{aligned} \tag{3.13}$$

(here $r^2 = r_\mu r^\mu$ etc.) Imposing a high-momentum cuttoff Λ, we see that although the first term has a divergence proportional to Λ^{n+2}, the order k^2 term only has Λ^n. This example can be easily modified to show the anisotropic dressing of dispersion relations in the presence of high energy Lorentz violation: if we split space and time in the above integral, and add a high-energy Lorentz violating term to give

$$\int d^3r dr_0 \frac{r^n}{(r_0+k_0)^2 - (r+k)^2 + m^2 + \frac{1}{M^2}(r+k)^4} \tag{3.14}$$

Expanding to second order in k_0, k, we would see that the terms proportional to k_0^2 are the same as above (with a suitably modified denominator), whereas the terms proportional to k^2 now include terms from $(r+k)^4$; moreover, by dimensional analysis these terms likely carry the largest divergences.

Chapter 4

Lifshitz scaling in Quantum Field Theory

In this chapter we introduce Lifshitz theories, the class of Lorentz-violating theory that we shall be studying for the rest of this work. We examine the general concepts behind such models and some of the features that they generally exhibit. We discuss the motivations for studying these models and provide a worked example of the methods introduced in chapter 3 in the context of Lifshitz theories.

4.1 Introduction

Lifshitz models, which will be our primary focus here, were originally developed in the field of condensed matter [16], they posit anisotropic scaling between one space-time co-ordinate and the others; in high-energy physics, this distinguished co-ordinate is generally taken to be time. In its original context, the distinguished co-ordinate was a spatial direction in an anisotropic material; studies of similar high-energy models have been performed, mostly in $4+1$ dimensions with the anomalous space dimension being an "extra" dimension, but those models are not of interest to us here.

A Lifshitz-scaling model is assumed to be invariant, in the ultra-violet, under the re-scaling:

$$x_i \to a x_i,\ t \to a^z t \tag{4.1}$$

for some integer $z > 1$, the "critical exponent". We have imposed different "scaling dimensions" on the co-ordinates, so for consistency we take matching mass dimensions $[t] = z[x]$.

A simple Lifshitz model is given by the Lagrangian density, for some scalar ϕ

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi\partial_t\phi + (-1)^z\partial_i\phi(\Delta)^{z-1}\partial_i\phi) \tag{4.2}$$

We could then introduce relevant operators, constructed only from space derivatives of ϕ, to obtain

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi\partial_t\phi - M^{2(z-1)}\partial_i\phi\partial_i\phi + \ldots + (-1)^z\partial_i\phi(\Delta)^{z-1}\partial_i\phi) \ , \tag{4.3}$$

where the ellipsis represents possible intermediate terms for $z > 2$. This "breaks" the anisotropic rescaling, considered as a global symmetry.
A similar Lagrangian density for fermions would be something of the form

$$\mathcal{L}_{fermi} = \overline{\psi}i\gamma^0\dot{\psi} - \overline{\psi}iM^{z-1}\gamma^i\partial_i\psi - \ldots - \overline{\psi}(i\gamma^i\partial_i)^z\psi \ , \tag{4.4}$$

as we shall see in chapters 5 and 6. Similarly, for vector bosons, we would have

$$\mathcal{L}_{vect} = \frac{1}{4}\left(2F_{0i}F^{0i} - F_{ij}(M^{z-1} + \ldots + (-\Delta)^{z-1})F^{ij}\right) \ . \tag{4.5}$$

In each case, we have been forced to introduce a dimensionful scale M with the lower-order space derivatives. If we consider the dispersion relation for ϕ

$$\omega^2 = M^{2(z-1)}k^2 + \ldots + k^{2z}, \tag{4.6}$$

we see that Lifshitz scaling will only be restored asymptotically at large energies $k^2 \gg M^2$. Indeed, at low energies, $k^2 \ll M^2$, the model will (classically) appear relativistic: to recover the usual dispersion relation, we must perform a time rescaling: $\omega = M^{(z-1)}\omega'$, so that

$$\omega'^2 = k^2 + \ldots + \frac{k^{2z}}{M^{2(z-1)}}. \tag{4.7}$$

4.2 Mass and spacetime dimensions

We shall work here with *mass* dimensions and adopt the convention that $[x] = -1$ and hence $[t] = -z$. In order that the action remain dimensionless, now that $[d^3xdt] = -(3+z)$, we must also alter the mass dimensions of the various fields; for instance, the scalar ϕ in 4.3 would, in a Lorentz symmetric theory have a mass dimension of

$$[\phi_1] = 1, \tag{4.8}$$

whereas we now have

$$[\phi_z] = \frac{1}{2}(3-z). \tag{4.9}$$

Similarly, the mass dimensions of couplings must also change: we introduce an interaction term to the Lagrangian, say $g\phi^4$; in a relativistic theory we would have

$[g_1] = 0$ but in a general Lifshitz theory we have $[g_z] = 3(z-1)$. We see that a previously marginal coupling has become relevant for $z > 1$, this will be important in the next section, where we discuss the effects on renormalisability of Lifshitz scaling.

We note more generally that in d spatial dimensions, a scalar field has mass dimension $\frac{1}{2}(d-z)$ and so, at the critical value $d = z$, all possible polynomial interactions are marginal or relevant.

Example: Liouville-Lifshitz theory

This is demonstrated particularly well by the "Liouville-Lifshitz" theory of [17]. In a $z = d = 3$ Lifshitz scalar theory, the authors consider a model with an *exponential* potential:

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi\partial_t\phi - \partial_i\phi(\Delta)^2\partial_i\phi) - \frac{\mu^6}{g^2}e^{g\phi} \tag{4.10}$$

which has an infinite number of interactions, all of which are marginal.

This model has a number of remarkable properties: one is that, due to a symmetry of the theory ($\phi \to \phi - a$, $\mu \to \mu e^{ga}$), one can derive an extra constraint on the effective potential, showing it retains the exponential form of the bare potential.

Because of the improvements in renormalisability caused by Lifshitz scaling (that we shall discuss in the next section), there is only one source of divergences in this model, though it appears in infinitely many graphs due to the infinite number of interactions; nonetheless, one may calculate all counterterms in this theory by induction (on the number of loop integrals). One finds an eventual effective potential of

$$V(\phi_r) = \frac{\mu_r^2}{g_r^2}e^{g_r\phi} \tag{4.11}$$

where we use a subscript r to denote renormalised quantities, and g_r can be expressed exactly as $g_r^{-1} = g^{-1} + \frac{g}{48\pi^2}$.

Most interesting, for our purposes here, is that the infra-red Lorentzian kinetic term $\partial_i\phi\partial_i\phi$ is *generated dynamically*, by quantum effects, in a classically Lifshitz-scaling model.

4.3 Renormalisability

The most interesting property of Lifshitz theories, from the perspective of high-energy physics, is that they improve the renormalisability of many interactions: compare the propagator for a free scalar in a relativistic model

$$\frac{1}{\omega^2 - k^2} \tag{4.12}$$

to that of the scalar in the model above

$$\frac{1}{\omega^2 - M^{2z-2}k^2 - ... - k^{2z}}. \tag{4.13}$$

Clearly, substituting the latter for the former would significantly decrease the ultraviolet divergence of any loop integral (over ω, k) in which it appeared.

Indeed, the effect of introducing Lifshitz scaling on the perturbative renormalisability of simple models can be calculated fairly easily: a Feynman graph in a perturbation series from a relativistic theory (without derivative interactions) with L loops, I_b internal boson lines and I_f internal photon lines, in $d+1$ dimensions has a superficial degree of divergence given by

$$D_1 = (d+1)L - I_f - 2I_b, \tag{4.14}$$

whereas the "same" graph in a Lifshitz-scaling theory would have (bearing in mind that the integrals over the time components of loop momenta now contribute z)

$$D_z = (d+z)L - zI_f - 2zI_b. \tag{4.15}$$

As the number of internal lines will never be less than the number of loops (and should grow faster than it if there are only finitely many interactions), the degree of divergence will generally decrease with increasing z. In this calculation, we have neglected the possibility of derivative interactions, such as are studied in chapters 6 and 7. In addition, maintaining gauge invariance may necessitate introducing new interactions for $z \neq 1$, through higher powers of gauge covariant derivatives; this, and the increase in the contribution to D from loops, may cause the divergence of some *low-order* terms to worsen, even as the total number of divergent graphs decreases.

One cannot achieve the same effect by simply adding higher derivative terms to a relativistic, Lorentz-symmetric theory; if we introduce higher derivatives in both space and time, the propagator becomes something of the form (omitting possible lower-derivative terms)

$$\frac{1}{\omega^{2z} - k^{2z}} = \frac{1}{2\omega^z}\left(\frac{1}{\omega^z + k^z} - \frac{1}{\omega^z - k^z}\right) \tag{4.16}$$

which clearly has imaginary poles that will lead to ghosts and thus violate unitarity. Indeed, generically, a theory with an equation of motion that is greater than second order in time derivatives will exhibit an Ostrogradski instability [18].

The renormalisation of Lifshitz theories has been studied extensively, for example in [19].

4.4 Notable features of Lifshitz theories

Lifshitz scaling alters the mass dimensions of integrals over time; this can lead to the possibility of generating dynamical masses from interactions that would not be permitted to produce these in a relativistic theory (see Chapter 5 and [20]). If a Lifshitz model is to appear classically relativistic in the infra-red, a mass scale must be introduced (such as M in the toy models of the previous sections), this is the scale by which higher-derivative terms are suppressed at low energies but may also provide a mass-scale for other behaviours, such as the previously mentioned dynamical mass generation.

Like many of the models mentioned in chapter 2, Lifshitz scaling breaks Poincaré symmetry to $SO(3)$ and translations: there is no modified notion of a "boost" in a Lifshitz model and there is a preferred rest frame defined by the anomalously scaling time co-ordinate. This will be particularly important when we examine Hořava's Lifshitz-scaling model of gravity (which will be discussed in more detail in the next chapter); as it is a *local* symmetry that is broken there, which leads to an extra degree of freedom for the graviton.

The higher derivatives in space, but not time, associated with Lifshitz theories lead to exactly the modified light-cone effects discussed in generality in chapter 2: as the (classically Lorentz invariant) low-energy kinetic terms of the fields are dressed anisotropically by quantum corrections, and in ways dependent on their couplings to the other fields of the theory, different species can exhibit differing effective "speeds of light", even at low energies (see section 4.6). This can similarly affect relativistic fields coupled to Lifshitz-scaling fields.

4.5 Other motivations for the study of Lifshitz theories

The recent interest in this class of theory was prompted by Hořava's proposal for a Lifshitz-scaling theory of gravity [4], which we shall discuss in more detail in chapters 7 and 8.

Many other speculative theories have special cases or limits that exhibit Lifshitz-like behaviour: the cases of Causal Dynamical Triangulations and Rainbow Gravity have already been mentioned above; we shall see in chapter 7 that Einstein-Aether or khronometric models can be related to Hořava gravity [21] [22] and through that, the MOND theories those are often used to model. The model of "ghost inflation", in which a scalar field condenses in a non-stationary background [23], exhibits Lif-

shitz like dynamics and can in turn be related to khronometric models.
Lifshitz theories have also been used to apply the AdS/CFT correspondence to anisotropic theories [24]. For a general review of Lifshitz theories in particle physics, see [25].

4.6 A worked example for effective actions in Lifshitz theories

Here, we introduce a very simple interacting Lifshitz scaling model, and calculate its effective action at one loop, by the methods described in chapter 3.

4.6.1 Model

We consider the $z = 2$ Lifshitz-scaling scalar field theory with Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi\partial_t\phi - M^2\partial_i\phi\partial_i\phi + \partial_i\phi\Delta\partial_i\phi) + \lambda\phi^3 \tag{4.17}$$

where λ is a coupling constant.
To calculate the corrections effective action, at lowest order, we need to consider the term

$$\frac{\delta^2\mathcal{L}[\phi(x)]}{\delta\phi(x)\delta\phi(x')} = (-\partial_t^2 + M^2\Delta - \Delta^2 + 6\lambda\phi(x))\delta(x - x')\ , \tag{4.18}$$

We introduce the shorthand $D = \partial_t^2 - M^2\Delta + \Delta^2$.

4.6.2 Calculation

The first-order correction to the effective action is given by

$$\delta\Gamma = \frac{i}{2}\log\det[(D - 6\lambda\phi(x))\delta(x - x')] \tag{4.19}$$

We use some basic properties of logarithms and determinants to write this as a series in ϕ (or, equivalently, λ):

$$\begin{aligned}\log\det[-(D + 6\lambda\phi(x))\delta(x - x')] &= \mathrm{Tr}[\log(D) + \log(1 - 6\lambda D^{-1}.\phi)] \\ &= \mathrm{Tr}[\log(D) - 6\lambda D^{-1}.\phi - 18\lambda^2(D^{-1}.\phi)^2 \\ &+ O(\phi^3)]\ ,\end{aligned} \tag{4.20}$$

where we note that "." represents a functional *matrix multiplication* in the same sense as the trace, that is

$$[A.B](u, v) = \int dw A(u, w)B(w, v) \tag{4.21}$$

where we have taken functions of one variable $f(x)$ to be diagonal as $f(x)\delta(x-y)$.

The first term in the expansion above is constant in ϕ and so has no physical effect, the second is a tadpole and will be removed by counterterms. It is the order ϕ^2 and higher terms that are of interest here (for this worked example, we shall restrict our attention to the lowest such).

We seek to evaluate $\mathrm{Tr}[18\lambda^2(D^{-1}.\phi)^2]$, this is most easily done in Fourier components. Firstly, we shall calculate the correction to the effective potential (i.e. the effective mass, as we work at order ϕ^2 here), this is straightforward: one takes the classical ϕ as constant, and so has

$$-2\phi^2\delta m^2 = \mathrm{Tr}[18\lambda^2(D^{-1}.\phi)^2] = \int \frac{d^3r dr_0}{(2\pi)^4} 18\lambda^2 \frac{1}{(r_0^2+M^2r^2+r^4)^2}\phi^2 \tag{4.22}$$

which would, were it not for the infra-red divergence caused by the lack of masses in this model, be a convergent integral (having a divergence of Λ^{-3} in some UV cutoff Λ, by power counting).

Calculating the lowest-order dispersion relation corrections is more involved if approached this way, in Fourier space terms, we seek the terms from $\tilde{\phi}(k)$ proportional to $k^2\tilde{\phi}(k)^2$, we must, therefore, assume a plane wave form for $\phi(x)$, such as $\phi_0 e^{-ik.x}$. It is generally more straightforward to approach this problem from a Feynman diagram perspective; $\Gamma[\phi]$ is the generating functional of one-particle irreducible diagrams and there is only one such two-legged diagram at one loop (or order λ^2) in this theory, one can simply introduce a non-vanishing external momentum to this graph. Either way, this leads to powers of k as well as the loop momentum r in the integral above; dropping the coefficients for clarity, we have

$$\begin{aligned}
& \int \quad d^3r dr_0 \frac{1}{(r_0^2+M^2r^2+r^4)((r_0-k_0)^2+M^2(r-k)^2+((r-k)^2)^2)} \\
= & \int \quad d^3r dr_0 \frac{1}{(r_0^2+M^2r^2+r^4)^2} + k_0^2\left[\frac{-1}{(r_0^2+M^2r^2+r^4)^3} + \frac{4r_0^2}{(r_0^2+M^2r^2+r^4)^4}\right] \\
& + \quad \left[\frac{-4(r.k)^2-(M^2+2r^2)k^2}{(r_0^2+M^2r^2+r^4)^3} + \frac{(2M^2+4r^2)^2(r.k)^2}{(r_0^2+M^2r^2+r^4)^4}\right] \\
& + \quad O(k^3).
\end{aligned} \tag{4.23}$$

This clearly demonstrates two important points: firstly, as expected, the terms proportional to k_0^2 and those proportional to $k^2 = k_i k_i$ are different, therefore the quadratic space- and time-derivative terms in the Lagrangian will be dressed differently. Of course, as there is only one species in this model, the effect of this anisotropic dressing can be removed by an appropriate space or time rescaling.

Secondly, observe that the terms proportional to k_0^2, k^2 are less divergent than those of the mass correction term above: Λ^{-7} and Λ^{-5} respectively (in terms of some cutoff), in this case, giving the expected 2 powers of momentum removed with k^2 as explained in chapter 3, but 4 powers of momentum removed with k_0^2, due to the anomalous scaling.

Chapter 5

Higher-order corrections in a four-fermion Lifshitz model

5.1 Introduction

This chapter is based on a paper [1], written with Jean Alexandre and Nicholas Houston.

In this chapter, we consider a Lifshitz-type four-fermion interaction model, which, in 3 space dimensions and for an anisotropic scaling $z = 3$, is renormalisable [19], unlike the equivalent Lorentz-symmetric model. Such models have previously been studied in [26], where two interesting properties were shown: dynamical mass generation and asymptotic freedom. This model serves as a good example of the features discussed in the previous sections: firstly, classical corrections to propagation relations are negligible while quantum corrections are much more significant; also, divergences in high order graphs are lessened.

Indeed, the overall superficial degree of divergence of the graphs of the theory is $D = 6 - 3E/2$, where E is the number of external lines, compared to the $D = 4 - 3E/2 + 2n$ of the Lorentz symmetric case, where n is the number of vertices in the graph.

If one considers the propagator ($E = 2$), the corresponding corrections have a superficial degree of divergence equal to 3, the best (non-trivial) case in the Lorentz-symmetric model. The coefficient of the cubic divergence may cancel for some graphs, but we calculate here a two-loop graph which shows that the divergence in the model is at least quadratic. Therefore, although renormalisable, this Lifshitz model still contains "large" divergences.

Our model features two massless fermion flavours, coupled with four-fermion interactions which do not respect flavour symmetry. After showing the occurrence of dynamical flavour oscillations in this model, we calculate the modified dispersion relations for these two fermions, arising from quantum fluctuations. Classically, all fermions have the same dispersion relations, with higher order powers of the space momentum $\vec{p}$, suppressed by the large mass M, typical to these models. These dispersion relations coincide with the expected Lorentz-invariant one in the infrared (IR) regime $|\vec{p}| << M$. Taking into account quantum corrections, though, modifies this IR limit: it is known in Lifshitz-type studies that different species of particles see different effective light cones (see chapter 2 or [5]). Since our model breaks flavour symmetry, the dressed IR dispersion relations are different from the Lorentz-invariant one, and we show that the corresponding corrections are quadratically divergent, and furthermore too significant to be taken to represent any physical effect. As a consequence, a proper treatment of the model would involve renormalisation; defining counterterms to absorb these divergences, introducing new parameters and so no prediction can be made as far as Lorentz-violating propagation is concerned.

The next section derives the dynamical masses for the model, including the mass mixing terms necessary for flavour oscillations. From our study of dynamically induced flavour oscillations, and taking into account experimental data on neutrino oscillations, we derive values for the coupling constants of our model, which, as expected, are perturbative. Although neutrinos are not Dirac fermions, the corresponding experimental constraints give a good order of magnitude for the parameters in our model. In the third section, we demonstrate the asymptotic freedom of the model, based on a one-loop calculation. The four-point function has a vanishing superficial degree of divergence, such that higher order corrections cannot change the sign of these beta functions. The effective IR dispersion relations, dressed by quantum fluctuations, are derived in section 4. For this, we need to go to two loops, since the one-loop correction to the fermion propagators is momentum-independent. Detailed calculations are given in the Appendix A, where we perform part of the integration analytically and then integrate the rest numerically.

5.2 Flavour oscillations

In this section, we first introduce the concept of neutrino flavour oscillations and then describe how flavour oscillations can be generated dynamically, along with masses, from flavour-mixing interactions, as suggested in [27]. We use these expressions for the dynamical masses, together with experimental data, to constrain the values of the coupling constants of our model.

Oscillations of massless neutrinos are studied in [28], where neutrinos are considered open systems, interacting with an environment. Such oscillations have also

been studied in [29], in the framework of Lorentz-violating models, involving non-vanishing vacuum expectation values for vectors and tensors. Whilst these studies have been questioned by phenomenological constraints [30], our present model, based on anisotropic space time and higher order space derivatives, is not excluded.
Flavour oscillations were also related to superluminality in [31], where it is shown that, if superluminality is due to a tachyonic mode, the latter can be stabilised by flavour mixing. Finally, in [32], superluminal effects are related to the extension of a single neutrino wave function, where the oscillation mechanism plays a role in the uncertainty of the neutrino position.

5.2.1 Background: mechanisms for generating neutrino masses and oscillations

Neutrino oscillations have been unambiguously observed, thus implying that the flavour operator of the neutrino is not simultaneously diagonalisable with their mass operator. This in turn implies that the neutrino species must have masses; however, these masses are far smaller than those of any other massive standard model fermions. This suggests that some mechanism other than the usual interaction with the Higgs field, which generally leads to masses of the same order as the Higgs vacuum expectation.

Seesaw mechanism

One simple suggested solution to this problem is the so-called "seesaw" mechanism, most simply explained here in the case of a single neutrino flavour: we assume this neutrino, ν_α couples with an uncharged ("sterile") right-handed spinor, η^α through the usual mass terms:

$$m\nu_\alpha\eta^\alpha \tag{5.1}$$

where m is assumed to be of the same order as other standard model fermion masses. However, unlike ν, no symmetry protects η from having a Majorana mass term:

$$\frac{1}{2}M\eta_a\eta^a \tag{5.2}$$

which could, in principle, be very large (perhaps around the GUT scale, if symmetry breaking there generates it); we assume that $M \gg m$. If both these mass terms are non-zero, the mass and flavour eigenstates are no-longer identical (though they will be very close); diagonalising the mass matrix gives one very heavy fermion with mass

$$m_h \sim M \tag{5.3}$$

which is mostly sterile and one much lighter fermion with a mass

$$m_l \sim \frac{m^2}{M} \ll m \tag{5.4}$$

which is assumed to be the observed neutrino.

While this model is simple and intuitive, it posits extra particles of which we have no evidence; in this chapter, we examine an alternative: that the neutrinos gain mass dynamically, through some non-perturbative process.

Dynamical mass generation

The dynamical generation of fermion masses, through quantum corrections to a massless model is hardly a novel idea: the Nambu-Jona-Lasinio model (see [33] for a review) was at one point considered as a possible origin of quark masses, before more renormalisable alternatives were found. More recently, much work has been done on the generation of masses through spontaneous chiral symmetry breaking in the presence of a constant magnetic field [34].

What dynamical mass models generally have in common is

- the introduction, often "by hand" of a mass scale, generally a cutoff. Though it may be possible to take a "simultaneous" limit of this scale and the couplings, such that the dynamical mass remains finite and non-zero.
- The dynamical masses are generally non-analytic in the theory's coupling constants, meaning that the masses could not be found by standard perturbation theory, and require the use of non-perturbative approaches, such as the Schwinger-Dyson equation, or alternative methods of expansion (such as large N). An exception to this tendency is the "quasi-relativistic" model studied in [35].

In our particular model, we introduce a four-point interaction between two massless active neutrino species, such an interaction would be non-renormalisable in a relativistic model, but we shall work in a $z = 3$ Lifshitz theory. We demonstrate that masses are dynamically generated and that those masses may take experimentally allowed values with reasonable values of the couplings. The interaction we introduce mixes flavours, and the dynamical masses inherit this, thus also dynamically generating flavour oscillations.

5.2.2 Flavour symmetry violating 4-fermion interactions

In a $z = 3$ Lifshitz theory, in $d = 3$ space dimensions, we consider two flavours of massless Dirac fermions ψ_1, ψ_2, and the free action

$$S_{free} = \int dtd^3x \left(\overline{\psi}_a i\gamma^0 \dot{\psi}_a - \overline{\psi}_a (M^2 - \Delta)(i\vec{\partial} \cdot \vec{\gamma})\psi_a \right) \qquad a = 1, 2 \ , \tag{5.5}$$

where $[M] = 1$ and $[\psi_a] = 3/2$, and a dot over a field represents a time derivative. For the dispersion relations to be consistent in the IR (see eq.(5.8) below), one can consider M typically of the order of a Grand Unified Theory scale (GUT), although we will show that our results only slightly depend on the actual value of M. We introduce the following renormalisable, flavour-violating and attractive 4-fermion interactions

$$S_{int} = \int dtd^3x \Big(g_1\overline{\psi}_1\psi_1 + g_2\overline{\psi}_2\psi_2 + h(\overline{\psi}_1\psi_2 + \overline{\psi}_2\psi_1) \Big)^2 \ , \tag{5.6}$$

where the coupling constants g_1, g_2, h are dimensionless. As shown in [26], this kind of model exhibits dynamical mass generation, which can be seen only with a non-perturbative approach, as will be shown in the next section. Taking into account the dynamical masses, but ignoring quantum corrections to the kinetic terms, the dispersion relations are of the form

$$\omega^2 = m_{dyn}^6 + (M^2 + p^2)^2 p^2 \ , \qquad a = 1, 2 \ , \tag{5.7}$$

which, after the rescaling $\omega = M^2\tilde{\omega}$, leads to

$$\tilde{\omega}^2 = \tilde{m}_{dyn}^2 + p^2 + \frac{2p^4}{M^2} + \frac{p^6}{M^4} \ , \tag{5.8}$$

where $\tilde{m}_{dyn} = m_{dyn}^3/M^2$. One can see then that Lorentz-like kinematics are recovered in the IR regime $p^2 << M^2$, as expected in the framework of Lifshitz models. After the rescaling $t = \tilde{t}/M^2$, the action reads

$$\begin{aligned} S \;\; &= \;\; \int d\tilde{t}d^3x \Bigg(\overline{\psi}_a i \;\; \partial\!\!\!/ \psi_a + \overline{\psi}_a \frac{\Delta}{M^2}(i\vec{\partial} \cdot \vec{\gamma})\psi_a \\ & \qquad + \left[\frac{g_1}{M}\overline{\psi}_1\psi_1 + \frac{g_2}{M}\overline{\psi}_2\psi_2 + \frac{h}{M}(\overline{\psi}_1\psi_2 + \overline{\psi}_2\psi_1) \right]^2 \Bigg) \ , \end{aligned} \tag{5.9}$$

where we can see that the four fermion couplings $(g_a/M)^2$, $g_a h/M^2$ and $(h/M)^2$ are very small compared to the Fermi coupling $\simeq 10^{-5}$ GeV^{-2}, if M is of the order of a GUT scale, or even several orders of magnitude smaller, and g_a, h are perturbative. Finally, note that, for Large Hadron Collider energies up to few TeVs, the classical Lifshitz corrections p^4/M^2 and p^6/M^4 in the dispersion relation (5.8) are not detectable, if M is of the order of a GUT scale. For this reason, if one wishes to describe measurable non-relativistic effects in Lifshitz theories, these should be sought in quantum corrections to the IR dispersion relation, as we shall do here.

5.2.3 Superficial degree of divergence

It is interesting to note that, although this Lifshitz model has fewer divergences than the Lorentz-invariant ψ^4 theory, ultimately making it renormalisable, low order quantum corrections actually do not "behave better" than those in the Lorentz-invariant ψ^4 theory, since the superficial degree of divergence of the propagator is 3. To show this, we calculate via the usual approach the degree of divergence D of a graph with E external lines. Each loop gives an integration measure $dp_0 d^3p$, which has mass dimension 6, and each propagator has mass dimension -3. For a graph with I internal lines and L loops, the superficial degree of divergence is therefore $D = 6L - 3I$. As usual, because of momentum conservation, we also have $L = I - n + 1$, where n is the number of vertices of the graph. Finally, since we have 4-leg vertices, we also have the relation $4n = E + 2I$. Taking into account these constraints, we find $D = 6 - 3E/2$.
From this result, we see that the four-point function is at most logarithmically divergent, but the propagator has a superficial degree of divergence equal to 3, although the one-loop mass corrections are logarithmically divergent only, as we show in the next subsection.

5.2.4 Dynamical generation of masses

We now calculate the dynamical masses generated by the interaction (5.6). For this, we introduce the auxiliary scalar field ϕ to express the interaction as

$$\begin{aligned} \exp(iS_{int}) &= \int \mathcal{D}[\phi] \exp(iS_\phi) \ , \\ \text{with} \qquad S_\phi &= \int dtd^3x \left(-\phi^2 + 2\phi(g_1\overline{\psi}_1\psi_1 + g_2\overline{\psi}_2\psi_2 + h(\overline{\psi}_1\psi_2 + \overline{\psi}_2\psi_1)\right) \ , \end{aligned} \tag{5.10}$$

and we note that completeing the square above allows the ϕ path integral to be performed exactly, recovering our orignial action. We then calculate the effective potential for ϕ=constant as

$$\exp(i\mathcal{V}V_{eff}(\phi)) = \int \mathcal{D}[\psi_1, \overline{\psi}_1, \psi_2, \overline{\psi}_2] \exp(iS_{free} + iS_\phi) \ , \tag{5.11}$$

where $\mathcal{V}$ is the space time volume. This integration can also be done exactly, since $S_{free} + S_\phi$ is quadratic in fermion fields, and leads to an effective potential for ϕ. From the dispersion relation (5.8), one can see that a non-trivial minimum ϕ_{min} for this effective potential will give the flavour mixing mass matrix

$$\begin{pmatrix} m_1^3 & \mu^3 \\ \mu^3 & m_2^3 \end{pmatrix} = 2\phi_{min} \begin{pmatrix} g_1 & h \\ h & g_2 \end{pmatrix} \ , \tag{5.12}$$

which leads to the rescaled masses

$$\begin{pmatrix} \tilde{m}_1 & \tilde{\mu} \\ \tilde{\mu} & \tilde{m}_2 \end{pmatrix} = 2\frac{\phi_{min}}{M^2}\begin{pmatrix} g_1 & h \\ h & g_2 \end{pmatrix} . \tag{5.13}$$

As a consequence, the mass eigenstates are

$$m_{\pm} = \frac{\phi_{min}}{M^2}\left(g_1 + g_2 \pm (g_1 - g_2)\sqrt{1 + \tan^2(2\theta)}\right) , \tag{5.14}$$

where the mixing angle θ is defined by

$$\tan(2\theta) \equiv \frac{2h}{g_1 - g_2} . \tag{5.15}$$

With the auxiliary field, the Lagrangian can then be written in the form $\overline{\Psi}\mathcal{O}\Psi$, where

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} , \tag{5.16}$$

and the operator $\mathcal{O}$ is

$$\mathcal{O} = \begin{pmatrix} i\gamma^0\partial_0 - (M^2 - \Delta)(i\vec{\partial}\cdot\vec{\gamma}) + 2g_1\phi & 2h\phi \\ 2h\phi & i\gamma^0\partial_0 - (M^2 - \Delta)(i\vec{\partial}\cdot\vec{\gamma}) + 2g_2\phi \end{pmatrix} . \tag{5.17}$$

Integration over the fermions then gives the following effective potential for ϕ (where the Euclidean metric is used for the loop momentum)

$$\begin{aligned} V_{eff}(\phi) &= \phi^2 - \frac{1}{2}\int \frac{d\omega}{2\pi}\frac{d^3p}{(2\pi)^3}\ln\Bigg(\Big[\omega^2 + (M^2+p^2)^2p^2\Big]^2 \\ &\quad + 4\phi^2\Big[\omega^2 + (M^2+p^2)^2p^2\Big](g_1^2 + g_2^2 + 2h^2) + 16(g_1g_2 - h^2)^2\phi^4\Bigg) . \end{aligned} \tag{5.18}$$

A derivative with respect to ϕ gives

$$\frac{dV_{eff}}{d\phi} = 2\phi - \phi\int\frac{d\omega}{2\pi}\frac{d^3p}{(2\pi)^3}\frac{A\omega^2 + B}{(\omega^2 + C_+)(\omega^2 + C_-)} , \tag{5.19}$$

where

$$A = 4(g_1^2 + g_2^2 + 2h^2) \tag{5.20}$$

$$\begin{aligned} B &= 4(M^2+p^2)^2p^2(g_1^2 + g_2^2 + 2h^2) + 32\phi^2(g_1g_2 - h^2)^2 \\ C_{\pm} &= (M^2+p^2)^2p^2 + 2\phi^2\left[(g_1^2 + g_2^2 + 2h^2) \pm \sqrt{(g_1^2 - g_2^2)^2 + 4h^2(g_1+g_2)^2}\right] . \end{aligned} \tag{5.21}$$

The integration over frequencies ω leads to

$$\frac{dV_{eff}}{d\phi} = 2\phi - \phi\frac{1}{(2\pi)^2}\int_0^{\Lambda} p^2dp\left(\frac{B}{C_+\sqrt{C_-} + C_-\sqrt{C_+}} + \frac{A}{\sqrt{C_+} + \sqrt{C_-}}\right) \tag{5.22}$$

where Λ is the UV cut off, assumed to be large compared to M. A non-trivial minimum $\phi_{min} \neq 0$ for this effective potential is solution of the equation

$$8\pi^2 = \int_0^\Lambda p^2 dp \left(\frac{B}{C_+\sqrt{C_-} + C_-\sqrt{C_+}} + \frac{A}{\sqrt{C_+} + \sqrt{C_-}} \right) . \tag{5.23}$$

The dominant contribution of these logarithmically divergent integrals comes form $p \to \Lambda$, so we can therefore approximate

$$C_- \simeq C_+ \simeq p^6 + \frac{A}{2}\phi^2 \quad \text{and} \quad B \simeq A p^6 , \tag{5.24}$$

such that

$$\int_0^\Lambda p^2 dp \left(\frac{B}{C_+\sqrt{C_-} + C_-\sqrt{C_+}} + \frac{A}{\sqrt{C_+} + \sqrt{C_-}} \right) \simeq \frac{A}{3} \ln \left(\frac{2\sqrt{2}\Lambda^3}{\phi\sqrt{A}} \right) . \tag{5.25}$$

The gap equation (5.23) then gives

$$\phi_{min} \simeq \frac{\Lambda^3}{\sqrt{g_1^2 + g_2^2 + 2h^2}} \exp\left(\frac{-6\pi^2}{g_1^2 + g_2^2 + 2h^2} \right) , \tag{5.26}$$

and the rescaled masses (5.13) are

$$\tilde{m}_a \simeq \frac{2g_a}{\sqrt{g_1^2 + g_2^2 + 2h^2}} \frac{\Lambda^3}{M^2} \exp\left(\frac{-6\pi^2}{g_1^2 + g_2^2 + 2h^2} \right) \tag{5.27}$$

$$\tilde{\mu} \simeq \frac{2h}{\sqrt{g_1^2 + g_2^2 + 2h^2}} \frac{\Lambda^3}{M^2} \exp\left(\frac{-6\pi^2}{g_1^2 + g_2^2 + 2h^2} \right) . \tag{5.28}$$

As expected, these masses are not analytical in the coupling constants and could not have been obtained with a perturbative expansion. Similar results have been obtained in the context of magnetic catalysis [34], based on the Schwinger-Dyson approach, and also for Lorentz-violating extensions of QED [36]; neither of these studies, however, feature the anisotropic space time studied here.

Finally, we note that the approach adopted here, based on an effective potential for the auxiliary field ϕ, is in principle valid for a large number of flavours. This auxiliary field depends on space and time, and its fluctuations around the minimum ϕ_{min} may induce new fermion interactions which we have neglected here. For N fermion flavours, assuming the couplings scale as $g^2 \sim 1/N$, these fluctuations are suppressed by $1/N$, which justifies the approach (see [37] for detailed calculations in the scalar case). In our case, $N = 2$ is not "large", but the corresponding order of magnitude for the dynamical masses is sufficient for a suitably accurate determination of the coupling constants g_1, g_2, as explained in the next subsection.

Comparison to the Nambu-Jona-Lasinio model

A brief comparison should be made here to the superficially similar Nambu-Jona-Lasinio (NJL) model, which also exhibits dynamical mass generation through a similar mechanism by coupling through a four-point current term of the form

$$\mathcal{L}_{NJL} = \overline{\psi} i\gamma^\mu \partial_\mu \psi + \frac{G}{4}[(\overline{\psi}\psi)^2 - (\overline{\psi}\gamma^5\psi)], \tag{5.29}$$

where a sum over flavours is implicit (see [33] for a comparatively recent review). Similarly to our model, auxiliary fields may be introduced (two, in this case, though only one will have a non-zero vacuum expectation) to make the action quadratic in the fermion fields and a dynamical mass calculated; however, the NJL model exhibits a feature that our model lacks: there is a *critical value* for the coupling, below which no dynamical masses are generated; said value depends inversely on the size of the cutoff, so could be considered an artefact of the regularisation. However, the NJL model is also non-renormalisable in isotropic spacetime and so should be considered effective, and the cutoff indicative of some scale of new physics. By contrast, our dynamical masses seem to exist for arbitrarily small values of the couplings.

5.2.5 Experimental constraints

From the expressions (5.14), we obtain the following difference of squared mass eigenstates

$$\begin{aligned}\Delta m^2 &= \frac{4}{\cos(2\theta)} \frac{g_1^2 - g_2^2}{g_1^2 + g_2^2 + \tan^2(2\theta)(g_1 - g_2)^2/2} \\ &\qquad \times \frac{\Lambda^6}{M^4} \exp\left(\frac{-12\pi^2}{g_1^2 + g_2^2 + \tan^2(2\theta)(g_1 - g_2)^2/2} \right) .\end{aligned} \tag{5.30}$$

Experimental constraints are [38]

$$\begin{aligned}\Delta m_{12}^2 &= 7.59(7.22 - 8.03) \times 10^{-5} \ (\text{eV})^2 \\ \sin^2\theta_{12} &= 0.318(0.29 - 0.36) \ ,\end{aligned} \tag{5.31}$$

and we plot in fig.(5.1), from the expression (5.30), the set of points in the plane g_1, g_2 which are allowed, given the experimental constraints (5.31). We consider $\Lambda \simeq 10^{19}$ GeV, corresponding to the Planck mass. An important property is that the result is not very sensitive to the value of the mass scale M: because of the exponential dependence in eq.(5.30), an increase of several orders of magnitude in M leads to an increase of just a few percent for the couplings g_a, as shown in the following table. We consider the situation where $h << 1$, such that $g_1 \simeq g_2$, according to eq.(5.15). We tabulate the approximately common value for the coupling constants as a function of the ratio M/Λ:

M/Λ	10^{-16}	10^{-15}	10^{-14}	10^{-13}	10^{-12}	10^{-11}
$g_1 \simeq g_2$	0.46	0.47	0.48	0.48	0.49	0.50
M/Λ	10^{-10}	10^{-9}	10^{-8}	10^{-7}	10^{-6}	10^{-5}
$g_1 \simeq g_2$	0.51	0.53	0.54	0.55	0.56	0.58

Table 5.1: Coupling constants for different mass scales M, when $h << 1$ and $\Lambda = 10^{19}$GeV.

On fig.(5.1), the thin line represents the set of points satisfying the constraint

$$\left| \ln \left(\frac{\Delta m^2_{experimental}}{\Delta m^2_{calculated}} \right) \right| \leq 1 \ , \tag{5.32}$$

and the thick line represents the set of points such that the largest mass eigenvalue is between 10^{-3} and 1 eV. We see that the coupling constants appearing in the theory are then of the order $g_a^2 \simeq 0.25$, and can be considered perturbative.

5.3 Asymptotic freedom

We now calculate the one-loop coupling constants, for $h << 1$, and we show that the theory is asymptotically free. For simplicity, we set $h = 0$ but still keep $g_1 \neq g_2$. The bare interaction can be expressed as

$$g_1^2 \ (\overline{\psi}_1\psi_1)^2 + G \ \overline{\psi}_1\psi_1\overline{\psi}_2\psi_2 + g_2^2 \ (\overline{\psi}_2\psi_2)^2 \ , \qquad G = 2g_1g_2 \ , \tag{5.33}$$

and the dressed interaction is of the form

$$(g_1^2 + \delta g_1^2)(\overline{\psi}_1\psi_1)^2 + (G + \delta G)\overline{\psi}_1\psi_1\overline{\psi}_2\psi_2 + (g_2^2 + \delta g_2^2)(\overline{\psi}_2\psi_2)^2 \ . \tag{5.34}$$

Note that no symmetry imposes any relation between δG and $\delta g_1^2, \delta g_2^2$: the interaction $\overline{\psi}_1\psi_1\overline{\psi}_2\psi_2$ is dressed independently of the interactions $(\overline{\psi}_1\psi_1)^2$ and $(\overline{\psi}_2\psi_2)^2$.

5.3.1 One-loop Fermi coupling

If one denotes

$$\begin{aligned} N_a(\omega, \vec{p}) &= \omega\gamma^0 - (M^2 + p^2)(\vec{p} \cdot \vec{\gamma}) + m_a^3 \\ D_a(\omega, \vec{p}) &= \omega^2 - (M^2 + p^2)^2 p^2 - m_a^6 \ , \end{aligned} \tag{5.35}$$

the generic graph for the one-loop corrections is

$$I_{ab} = \int \frac{d\omega d^3p}{(2\pi)^4} \ \frac{iN_a(\omega, \vec{p}) iN_b(\omega, \vec{p})}{D_a(\omega, \vec{p}) D_b(\omega, \vec{p})}. \tag{5.36}$$

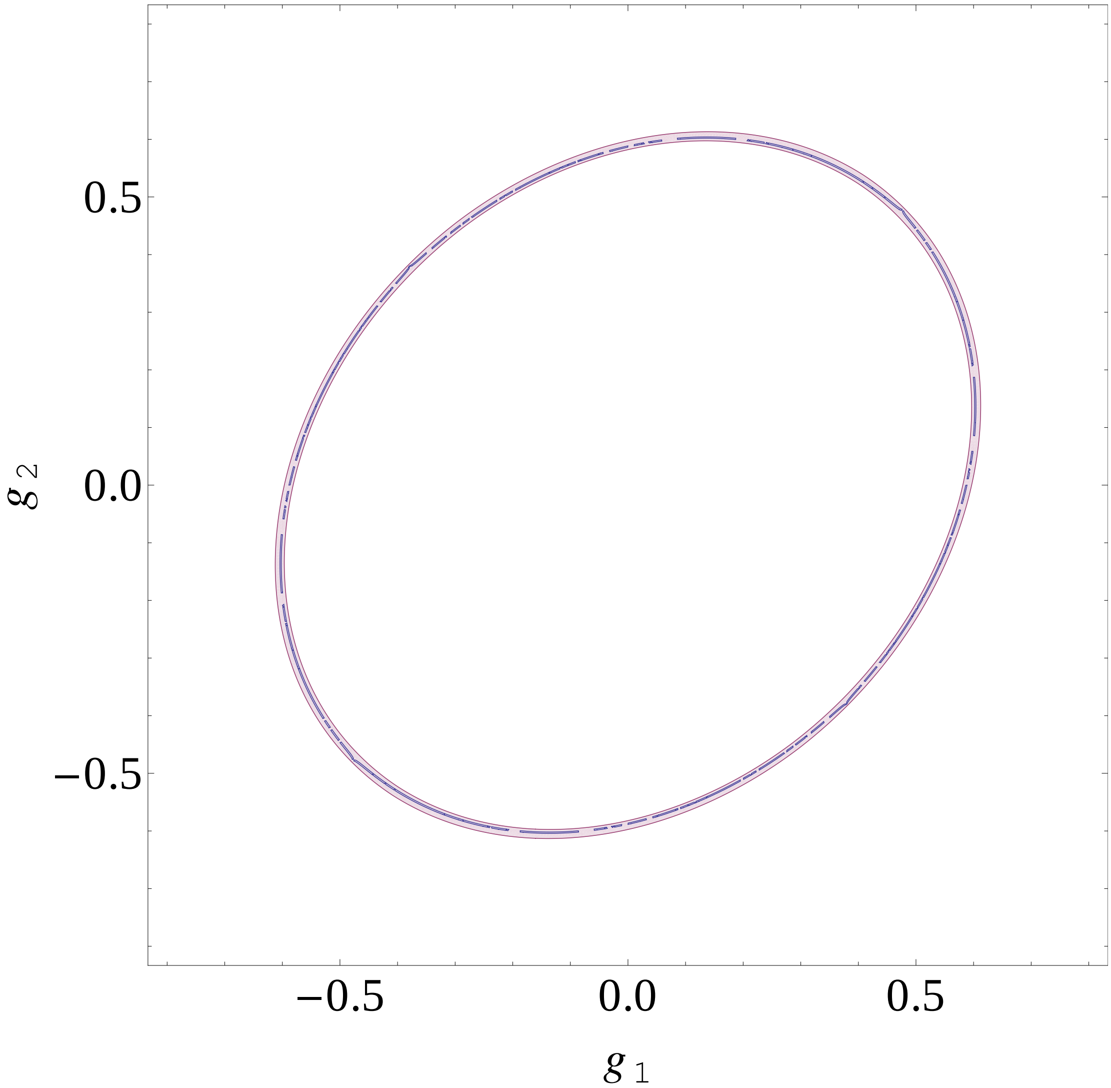

Figure 5.1: Values of g_1 (x-axis) and g_2 (y-axis) allowed by experimental constraints, for $M/\Lambda = 10^{-11}$. Negative values are allowed, since the physical quantities depend on the square of the coupling constants only. Points where $g_1 = g_2$ are strictly speaking not allowed, since at these points $\Delta m^2 = 0$. However, the resulting logarithmic singularity is very localised in the parameter space, such that we can safely chose g_1 and g_2 perturbatively close to each other.

When $\Lambda >> M$, we obtain

$$
\begin{aligned}
I_{ab} &= -\int \frac{d\omega d^3p}{(2\pi)^4}\left(\frac{m_a^3/(m_a^3-m_b^3)}{\omega^2-(M^2+p^2)^2p^2-m_a^6}-\frac{m_b^3/(m_a^3-m_b^3)}{\omega^2-(M^2+p^2)^2p^2-m_b^6}\right) \quad (5.37)\\
&= \frac{i}{4\pi^2(m_a^3-m_b^3)}\int_0^\Lambda p^2dp\left(\frac{m_a^3}{\sqrt{(M^2+p^2)^2p^2+m_a^6}}-\frac{m_b^3}{\sqrt{(M^2+p^2)^2p^2+m_b^6}}\right)\\
&\simeq \frac{i}{12\pi^2(m_a^3-m_b^3)}\int_0^{\Lambda^3} dx\left(\frac{m_a^3}{\sqrt{x^2+m_a^3}}-\frac{m_b^3}{\sqrt{x^2+m_b^3}}\right)\\
&\simeq \frac{i}{4\pi^2(m_a^3-m_b^3)}\left[m_a^3\ln\left(\frac{\Lambda}{m_a}\right)-m_b^3\ln\left(\frac{\Lambda}{m_b}\right)\right] .
\end{aligned}
$$

The integral (5.37) diverges logarithmically, unlike the Lorentz symmetric case where it diverges quadratically. Note that, when $m_b \to m_a$, the previous result is regular and leads to

$$I_{aa} \simeq \frac{i}{4\pi^2}\ln\left(\frac{\Lambda}{m_a}\right) . \quad (5.38)$$

In order to calculate the number of graphs (5.37) contributing to the coupling corrections, we introduce the auxiliary field σ (a convenient rescaling of ϕ from the previous section) and write the four-fermion interactions in the form

$$-\frac{1}{2}\sigma^2+\sigma\sqrt{2}(g_1\overline{\psi}_1\psi_1+g_2\overline{\psi}_2\psi_2) . \quad (5.39)$$

The scalar σ does not propagate, but is described by a fictitious propagator, which carries a factor of i. This propagator has to be understood in the limit where it shrinks to a point, leading to the fermion loops given by the expressions (5.37). The two vertices corresponding to the effective Yukawa interactions are $i\sqrt{2}g_1$ and $i\sqrt{2}g_2$.

The graphs corresponding to the four-point function are represented in fig.(5.2), in terms of the equivalent Yukawa interaction (5.39), where the last two graphs do not contribute to the four-fermion beta functions. Indeed, the general structure of the four point function is

$$\left\langle 0\left|\psi^\dagger\psi\psi^\dagger\psi\right|0\right\rangle = A\mathbf{1}\otimes\mathbf{1}+B_i\mathbf{1}\otimes\sigma^i+C_{ij}\sigma^i\otimes\sigma^j , \quad (5.40)$$

where the Dirac indices are omitted and the tensorial product allows for the two in and two out states. A is the only quantity contributing to the coupling constant, since the corresponding term has no Dirac index structure. The four-point function contains one divergence only, which is logarithmic, such that the divergent graphs are obtained only from the highest power of momentum in the numerator of propagators, i.e. from $\vec{p}\cdot\vec{\gamma}$ and not the mass term. The last two graphs of fig.(5.2) contain continuous lines of fermions with one internal propagator, such that the divergent part is contained in C_{ij} only. A is finite for these two graphs, and thus does not contribute to the beta function. More generally [39], to any order of the perturbation

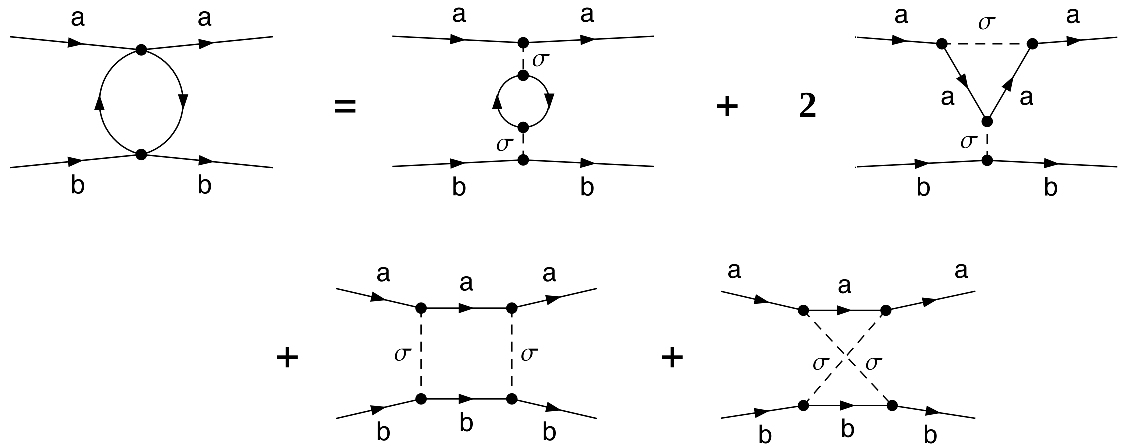

Figure 5.2: One-loop graphs involving the auxiliary scalar field, which contribute to the four-point function. Solid lines represent fermions and dashed lines represent the scalar. Only the first two diagrams, where the fermion lines cross an odd number of vertices, contribute to the four-fermion beta functions. The first graph corresponds to the insertion of a scalar self-energy, for each flavour, and involves a factor -4 for the trace over Dirac indices. The second graph has two contributions: one for each insertion of a vertex correction.

theory, any graph containing a open fermion line, which meets an even number of vertices, does not contribute to the beta functions of the model. One needs an even number of internal lines for the product of gamma matrices (appearing in $\vec{p}\cdot\vec{\gamma}$) to give a diverging term with a non-vanishing trace.

5.3.2 Beta-functions

The divergent one-loop correction to the four-fermion interactions are then given by the first two graphs of fig.(5.2), which are as follows.

- For the flavour preserving four-fermion interaction:
  (i) Graphs with the insertion of the one-loop scalar self-energy: both flavours contribute to the fermion loop, which induces a factor -4 for the trace over Dirac indices. The contribution is then,

$$-4i^2(i\sqrt{2}g_a)^4 I_{aa} - 4i^2(i\sqrt{2}g_a)^2(i\sqrt{2}g_b)^2 I_{bb} = 16g_a^2(g_a^2 I_{aa} + g_b^2 I_{bb}) \ . \qquad (5.41)$$

  (ii) Graphs with the insertion of the one-loop Yukawa interaction: only the flavour a plays a role, and the contribution is

$$2i^2(i\sqrt{2}g_a)^4 I_{aa} = -8g_a^4 I_{aa} \ . \qquad (5.42)$$

  The total contribution must be identified with the correction to the bare graph $i(i\sqrt{2}g_a)^2$, such that

$$i\delta g_a^2 = -4g_a^2(g_a^2 I_{aa} + 2g_b^2 I_{bb}) \ , \qquad (5.43)$$

and the corresponding beta function is therefore

$$\beta_a \equiv \Lambda \frac{\partial(\delta g_a^2)}{\partial \Lambda} = -\frac{g_a^2}{\pi^2}(g_a^2 + 2g_b^2) \ . \tag{5.44}$$

- For the flavour-changing interaction:

  (i) Graphs with the insertion of the one-loop scalar self-energy:

$$16 g_a g_b (g_a^2 I_{aa} + g_b^2 I_{bb}) \ ; \tag{5.45}$$

  (ii) Graphs with the insertion of the one-loop Yukawa interaction:

$$-4 g_b g_a^3 I_{aa} - 4 g_a g_b^3 I_{bb} \ ; \tag{5.46}$$

  The total contribution must be identified with the correction to the bare graph iG

$$i\delta G = -12 g_a g_b (g_a^2 I_{aa} + g_b^2 I_{bb}) \ , \tag{5.47}$$

  and the corresponding beta function is therefore

$$\beta_G = -3 \frac{g_a g_b}{\pi^2}(g_a^2 + g_b^2) \ . \tag{5.48}$$

One can conclude from this one-loop analysis that the theory is asymptotically free, since higher orders also diverge at most logarithmically, and cannot change the sign of the one-loop beta functions. Note that, when $g_1 = g_2$, then $\beta_G = 2\beta_a$, as expected from the $O(2)$ symmetry.

5.4 Two-loop propagator

Since Lifshitz theories explicitly break Lorentz symmetry, space and time derivatives are dressed differently by quantum corrections. As explained in chapter 2: if one considers only one particle, or several particles in a given flavour multiplet, frequency and space momentum can always be rescaled in such a way that the particles have the usual Lorentz-like IR dispersion relation (after neglecting the higher order powers of the space momentum, suppressed by M). However if one considers several particles without flavour symmetry, then it becomes necessary to perform a flavour-independent rescaling of frequency and space momentum, such that different particles see different effective light cones. This is the case we consider here.
The dispersion relations (5.8) are not modified at one loop, since the corresponding graphs do not depend on the external momentum. We therefore have to go to two loops to find the first quantum corrections, corresponding to the graphs represented on fig.(5.3), in terms of the equivalent Yukawa model (5.39).
We note here that the two-loop propagator is evaluated in [40] for a scalar ϕ^4 theory, in 6 spacial dimensions and for $z = 2$. This calculation is done in the massless case

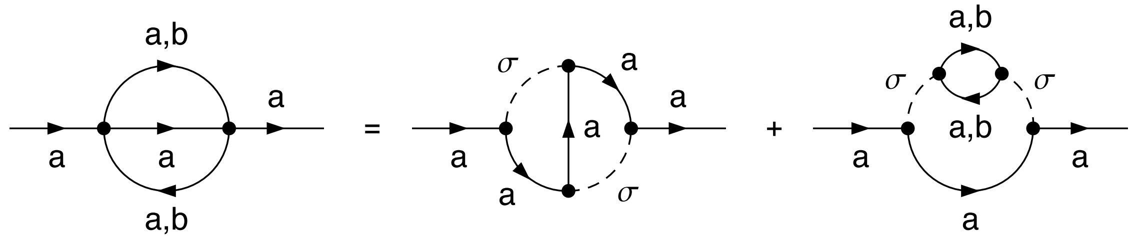

Figure 5.3: Two-loop contributions to the propagator. The fermion loop in the second graph involves a contribution from each flavour, and a factor -4 for the trace over Dirac indices.

and in the absence of quadratic space derivatives. Dimensional regularisation is used there, such that the power of the cut off does not appear explicitly in the results. The authors conclude that the the Lorentz-symmetry breaking terms flow to 0 in the deep IR.

5.4.1 Self energy

The perturbative graphs on fig. (5.3) can be calculated with massless bare propagators, since the two-loop graphs contain no IR divergence. As a consequence, these graphs are flavour independent (besides an overall factor depending on the coupling constants), and they involve the integrals

$$\begin{aligned} I(k_0,\vec{k}) &= i^2\int\frac{dp_0d^3p}{(2\pi)^4}\int\frac{dq_0d^3q}{(2\pi)^4}\;\frac{iN(-p)iN(-q)iN(p+q+k)}{D(-p)D(-q)D(p+q+k)} \\ J(k_0,\vec{k}) &= i^2\int\frac{dp_0d^3p}{(2\pi)^4}\int\frac{dq_0d^3q}{(2\pi)^4}\;\frac{\mathrm{Tr}[iN(-p)iN(-q)]iN(p+q+k)}{D(-p)D(-q)D(p+q+k)}\;, \end{aligned} \tag{5.49}$$

where the trace in J arises from the fermion loop, and the factors i^2 are for the scalar propagators. Taking into account the different possibilities for the self-energy of flavour a, we obtain

- $(i\sqrt{2}g_a)^4 I$ for the graph without a fermion loop;
- $[(i\sqrt{2}g_a)^4+(i\sqrt{2}g_a)^2(i\sqrt{2}g_b)^2]J$ for the graph with a fermion loop: one contribution for each flavour in the loop.

We calculate these integrals in the appendix A, where we see that the only role of the fermion loop is to give a factor -4 from the trace over Dirac indices. We therefore have $J=-4I$, and the total contribution to the momentum-dependent two-loop self energy $\Sigma_a(k_0,\vec{k})$ is given by

$$\begin{aligned} & -i\Sigma_a(k_0,\vec{k}) \\ =\;& -4g_a^2(3g_a^2+4g_b^2)I \\ =\;& -i4g_a^2(3g_a^2+4g_b^2)\int\frac{dp_0d^3p}{(2\pi)^4}\int\frac{dq_0d^3q}{(2\pi)^4}\;\frac{N(-p)N(-q)N(p+q+k)}{D(-p)D(-q)D(p+q+k)}\;. \end{aligned} \tag{5.50}$$

The bare inverse fermion propagator is

$$S_{bare}^{-1} = k_0\gamma^0 - M^2\vec{k}\cdot\vec{\gamma} + \cdots \ , \tag{5.51}$$

where dots represent higher orders in $\vec{k}$. We parametrise the dressed inverse propagator as

$$S_{dressed}^{-1} = -m_a^3 + (1 - Y_a)k_0\gamma^0 - (1 - Z_a)M^2\vec{k}\cdot\vec{\gamma} + \cdots \ , \tag{5.52}$$

such that the self energy is

$$\Sigma_a(k_0, \vec{k}) = S_{bare}^{-1} - S_{dressed}^{-1} = m_a^3 + Y_a k_0\gamma^0 - Z_a M^2\vec{k}\cdot\vec{\gamma} + \cdots \tag{5.53}$$

The integrals (5.49) should then be expanded in the external frequency k_0 and momentum $\vec{k}$ in order to find the corrections Y_a, Z_a. The perturbative k-independent mass correction m_a^3 will be disregarded, since it is small compared to the dynamical masses (non-analytic in the couplings) that have already been calculated nonperturbatively in a previous section.

5.4.2 Dressed dispersion relations

From the self energy (5.53), the IR dispersion relation for the flavour a is

$$(1 - Y_a)^2 k_0^2 = m_a^6 + M^4(1 - Z_a)^2 k^2 + \cdots \ , \tag{5.54}$$

where $k = |\vec{k}|$. If we assume that the two fermion flavours are to be coupled to other particles, then one needs a flavour-independent rescaling of the dispersion relation. $k_0 \to M^2\tilde{k}_0$ leads then to the following product of the phase and the group velocities, v_p and v_g respectively

$$v_a^2 \equiv v_p v_g = \frac{\tilde{k}_0}{k}\frac{\partial\tilde{k}_0}{\partial k} = 1 + 2(Y_a - Z_a) + \mathcal{O}(k/M)^2 \ . \tag{5.55}$$

We calculate Y_a and Z_a in an appendix, A, by expanding analytically the integral I to first order in k_0 and $\vec{k}\cdot\vec{\gamma}$, we find a quadratic divergence of the form ($\Lambda >> M$)

$$Y_a - Z_a \simeq 4\kappa\ g_a^2(3g_a^2 + 4g_b^2)\frac{\Lambda^2}{M^2} \ , \qquad \kappa \simeq -3.49 \times 10^{-5} \tag{5.56}$$

where $a \neq b$. This result shows that the present model is of limited use for the prediction of Lorentz-violating propagation. Indeed, with the values of $\Lambda/M, g_1, g_2$ shown in Table 1, the result (5.56) is not perturbative: one cannot then reasonably treat the cutoff as physical and so must absorb the quadratic divergence with counterterms, such that the renormalised value of $Y_a - Z_a$ needs to be fixed by experimental data. Therefore the model cannot predict quantitative deviations from Special Relativity at low energies.

If these corrections were logarithmic, one might be able to infer from our result

a cut-off-independent beta function for the effective maximum speed seen by the fermions, which could lead to "realistic" predictions on potential sub/super-luminal propagation.
Note that the rescaling of frequency which leads to the speed squared (5.55) does not make apparent the fact that, if flavour symmetry is exactly satisfied, then the IR dispersion relations are relativistic. If one ignores possible interactions with other particles, one can further rescale

$$k^2 = \tilde{k}^2 \frac{1-Y_1}{1-Z_1}\frac{1-Y_2}{1-Z_2} , \tag{5.57}$$

which leads to the following IR dispersion relations

$$\begin{aligned} \tilde{k}_0^2 &\simeq \tilde{m}_1^2 + (1+2\delta v)\tilde{k}^2 \\ \tilde{k}_0^2 &\simeq \tilde{m}_2^2 + (1-2\delta v)\tilde{k}^2 \\ \text{where } \delta v &= 6\kappa(g_1^4 - g_2^4)\frac{\Lambda^2}{M^2} . \end{aligned} \tag{5.58}$$

One can see here that the Lorentz-invariant IR dispersion relations are recovered when $g_1 = g_2$. But for $g_1 \neq g_2$, one would need the difference $|g_1^2 - g_2^2|$ to be proportional to M^2/Λ^2 in order to deal with realistic phenomenology. Taking the largest value $M/\Lambda \simeq 10^{-5}$ with $g_a \simeq 0.58$ from Table 1, the upper bound $\delta v \leq 2\times 10^{-9}$ given by the supernovae SN1987a data [41] gives

$$|g_1^2 - g_2^2| \leq \frac{\delta v\ M^2/\Lambda^2}{6\kappa(g_1^2+g_2^2)} \simeq 10^{-15} , \tag{5.59}$$

such that flavour symmetry would need to be almost exact, which, if one were to interpret as imposing $h \ll g_a$, could be seen as representing an unnatural fine tuning.

5.5 Conclusion

In this chapter, we have shown that flavour oscillations can be generated dynamically through four-fermion interactions between (bare-)massless fermions. We have exhibited a Lifshitz-type model in which these interactions lead to a renormalisable theory. The IR dispersion relations of this model are significantly not Lorentz invariant (in the absence of flavour symmetry): despite the altered kinematics being near undetectable classically, quantum corrections lead to very large IR effects. Indeed, these effects are too large for the model to be considered predictive, without a great deal of fine tuning.
We would therefore suggest that any similar model should have logarithmic divergences at worst, to be considered reasonable phenomenologically. This is the case, for example, of Lifshitz-type Yukawa models [42], where one-loop corrections to the fermion dispersion relations are finite. Also, Lifshitz-type extensions of gauge

theories, which are super-renormalisable in 3+1 dimensions and for $z = 3$, feature interesting properties, see [43] and the next chapter.

It should be noted that this study treated its theory as effective, and thus its regularising cut-off Λ as a possible physical constant. In the next chapter, we deal with a model with an exact $U(1)$ gauge symmetry, making this method of regularisation not physically reasonable, as it does not respect the symmetries of the theory.

Chapter 6

Fermion effective dispersion relation for $z = 2$ Lifshitz QED

This chapter is based on a paper [2], written with Jean Alexandre.

6.1 Introduction

In this chapter, we study consequences of Lorentz symmetry violation in a $z = 2$ Lifshitz extension of QED in 3+1 dimensions, and we again discuss the non-trivial effects of quantisation. Because of the specific power of space momentum in propagators of the model, dimensional regularisation leads to some unusual behaviour for loop integrals, which are finite even when the space dimension goes to 3, as discussed in the appendix C. We check the consistency of the approach by calculating the (vanishing) corrections to the photon mass and the IR-divergence-free corrections to the dispersion relation for massless fermions. Our aim here is to study the phenomenological viability of a Lifshitz QED model.

$z = 2$ Lifshitz QED is super-renormalisable, but still contains power counting diverging graphs. Nevertheless, the would-be divergent graphs we calculate here happen to be finite in $3 - \epsilon$ space dimensions, after integration over frequencies, even in the limit $\epsilon \to 0$. This special feature is a consequence of the specific powers of the momentum in this model, a phenomenon that is explained further in the appendix C. Dimensional regularisation is still needed though, in order to define the would-be ultraviolet (UV) diverging integrals. But because we have $z = 2$, the limit $\epsilon \to 0$ never leads to a pole of the Gamma function and the graphs are UV finite. The consistency of the approach is checked with the vanishing of the photon mass correction. Divergences in scalar Lifshitz models are discussed for different values of z and of space dimension in [44], where, in the case of scalar QED, the effective potential for the scalar field is finite for $z = 2$ and in 3 space dimensions.

Another feature related to dimensional regularisation of this model is the appearance of poles in ϵ, but due to IR divergences in the case of massless fermions. In order to calculate the integrals analytically, we will consider this massless limit, since we are interested in the effective fermion dispersion relation, which is independent of the mass anyway. As expected, the poles in ϵ generated by IR divergence cancel each other in the calculation of the effective "maximum" speed v.

Section 2 presents the model and its properties, from which the one-loop self energies and fermion dispersion relation are calculated in section 3, with details in the appendix B. We eventually find that the model predicts a too important deviation from IR relativistic kinematics at one-loop, if one identifies the dimensionless coupling with the fine structure constant, whereas the classical theory is totally justifiable in the IR regime.

A similar study has been done in [45], for a Lorentz-symmetry-violating extension of QED - not of a Lifshitz type though - where a rescaling of fields and spacetime coordinates, after one-loop corrections, leads to a usual relativistic IR dispersion relation for the photon but to a subluminal propagation for fermions.

6.2 Model

We consider the following Lagrangian density for a $z = 2$ Lifshitz QED model, with metric (1,-1,-1,-1),

$$\mathcal{L} = \frac{1}{2}F_{0i}F_{0i} - \frac{1}{4}F_{ij}(M^2 - \Delta)F_{ij} + \overline{\psi}(iD_0\gamma_0 - iMD_k\gamma_k - D_kD_k - m^2)\psi \tag{6.1}$$

Where $D_\mu = \partial_\mu + ieA_\mu$ and $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$. The mass dimensions are, in 3+1 dimensions,

$$[A_j] = \frac{1}{2} \quad , \quad [A_0] = [\psi] = \frac{3}{2} \quad , \quad [e] = \frac{1}{2} \quad , \quad [M] = [m] = 1\ . \tag{6.2}$$

The theory is invariant under the $U(1)$ gauge symmetry

$$\psi \to \psi e^{i\Lambda} \quad , \quad \overline{\psi} \to \overline{\psi}e^{-i\Lambda} \quad , \quad A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\Lambda\ , \tag{6.3}$$

and the classical dispersion relations are, for the photon and the fermion respectively,

$$\begin{aligned} k_0^2 &= M^2k^2 + k^4 \\ k_0^2 &= M^2k^2 + (k^2 + m^2)^2\ , \end{aligned} \tag{6.4}$$

where k_0 denotes the frequency and $k = \sqrt{k_ik_i}$ the space momentum. After the rescaling $k_0 \to Mk_0$, the dispersion relations become

$$\begin{aligned} k_0^2 &= k^2 + \frac{k^4}{M^2} \qquad \text{(photon)} \\ k_0^2 &= m_R^2 + (1+\eta)k^2 + \frac{k^4}{M^2} \qquad \text{(fermions)}, \end{aligned} \tag{6.5}$$

where the rescaled mass is $m_R \equiv m^2/M$ and $\eta \equiv 2m^2/M^2$.
To get a idea of the orders of magnitude, let us consider the electron mass $m_R \simeq 0.5$ MeV and the GUT scale $M \simeq 10^{16}$ GeV. We have then $m^2 \simeq 5 \times 10^{12}$ GeV^2 and $\eta \simeq 10^{-19}$, which is well within the Lorentz symmetry violation bounds [7]. Also, the corrections k^4/M^2 are by far currently not detectable, for energies at most $k \sim 10$ TeV. For this reason, the classical model can be realistic phenomenologically. But, as we will stress later in this chapter, quantum corrections completely change this picture. It should also be noted that, unlike the model in the previous chapter, the use of gauge covariant derivatives, D_μ means that the number and type of interaction terms is dependent on the critical exponent z.

6.2.1 Propagators and interactions

The fermion propagator is given by

$$G(p_0, p) = i\frac{p_0\gamma_0 - Mp_i\gamma_i + p^2 + m^2}{p_0^2 - M^2p^2 - (m^2 + p^2)^2} \tag{6.6}$$

In order to obtain a simple photon propagator, we impose the equivalent of the Feynman gauge condition in our anisotropic spacetime, which is

$$\partial_0 A_0 - (-\Delta + M^2)\partial_k A_k = 0 \ , \tag{6.7}$$

which leads to a non-local gauge-fixing term in the Lagrangian [27], but to a well-behaved photon propagator. The gauge fixing term is

$$\mathcal{L}_{GF} = -(\partial_0 A_0 - (-\Delta + M^2)\partial_i A_i)\frac{1}{2(-\Delta + M^2)}(\partial_0 A_0 - (-\Delta + M^2)\partial_j A_j), \tag{6.8}$$

though it can be written in a manifestly local form with auxiliary fields and is of mass dimension 5, as required. Throughout this chapter, we shall treat the A_0 and A_i lines in graphs separately, as they give different contributions; the photon propagator is then

$$D_{00}(k_0, k) = -i\frac{M^2 + k^2}{k_0^2 - M^2k^2 - k^4} \tag{6.9}$$

$$D_{ij}(k_0, k) = i\frac{\delta_{ij}}{k_0^2 - M^2k^2 - k^4} \tag{6.10}$$

with no off-diagonal components (before quantum corrections).
There are three vertices: the 3-point $\overline{\psi}A_i\psi$, contributing $-ie(\gamma_i M + 2p_i^{(\psi)} + p_i^{(A)})$; the 3-point $\overline{\psi}A_0\psi$, contributing $ie\gamma_0$ and the 4-point $A_iA_j\psi\overline{\psi}$, contributing $-2ie^2\delta_{ij}$.
From the expressions for the vertices and propagators given above, the superficial divergence for an arbitrary graph is found to be

$$D = 5L - 2I_{A_0} - 4I_{A_i} - 2I_\psi + N_{3i} \ , \tag{6.11}$$

where I_x is the number of internal lines of particle x, L the number of loops and N_{3i} the number of three-point spacial-photon vertices. With the standard relations $L = 1 + \Sigma_x I_x - \Sigma_a N_a$ and $(2I_x + E_x) = \Sigma_a n_a N_a$ (for each species x, where n_a is the number of x-lines on the vertex a) we can re-write the above in terms of external lines, E, as

$$D = 6 - E_{A_i} - 2E_{A_0} - 2E_\psi - L \ . \tag{6.12}$$

After quantum corrections we will rescale fields and coordinates such that the photon has the usual IR relativistic dispersion relation, and we will look at the consequence on the fermion dispersion relation. From eq.(6.12), the superficial degree of divergence of the fermion propagator is 1 at one loop, such that the divergence of the fermion wave function renormalisation is logarithmic.

6.2.2 Symmetries of the dressed propagators

As in usual QED, the polarisation tensor $\Pi_{\mu\nu}$ is transverse, since this property arises from the structure of the source terms in the partition function and is independent of the details of the kinetic term for the gauge field. The most general form for Π_{ij} is, at quadratic order in the momentum,

$$\Pi_{ij} = ZM^2(k_i k_j - k^2\delta_{ij}) + Wk_0^2\delta_{ij} \ , \tag{6.13}$$

where Z, W are dimensionless corrections to be calculated. Similarly, unitarity is ensured by the Ward identities, which still hold as gauge invariance is unbroken. Transversality $k^\mu\Pi_{\mu\nu} = 0$ implies then that the other components of the polarisation tensor are of the form

$$\Pi_{0i} = Wk_0k_i \quad , \quad \Pi_{00} = Wk^2 \ , \tag{6.14}$$

such that the photon dressed propagator $\mathcal{D}_{\mu\nu}$ satisfies

$$\begin{aligned}
\mathcal{D}^{-1}_{00}(k_0, k) &= D^{-1}_{00}(k_0, k) + Wk^2 \\
\mathcal{D}^{-1}_{ij}(k_0, k) &= D^{-1}_{ij}(k_0, k) + ZM^2(k_ik_j - k^2\delta_{ij}) + Wk_0^2\delta_{ij} \\
\mathcal{D}^{-1}_{0i}(k_0, k) &= Wk_0k_i \ .
\end{aligned}$$

The fermion dressed propagator is denoted, at the lowest order in momentum,

$$\mathcal{G}^{-1}(p_0, p) = G^{-1}(p_0, p) + \delta m^2 + Xp_0\gamma_0 - Yp_i\gamma_iM \ , \tag{6.15}$$

where X, Y are dimensionless corrections to be calculated.

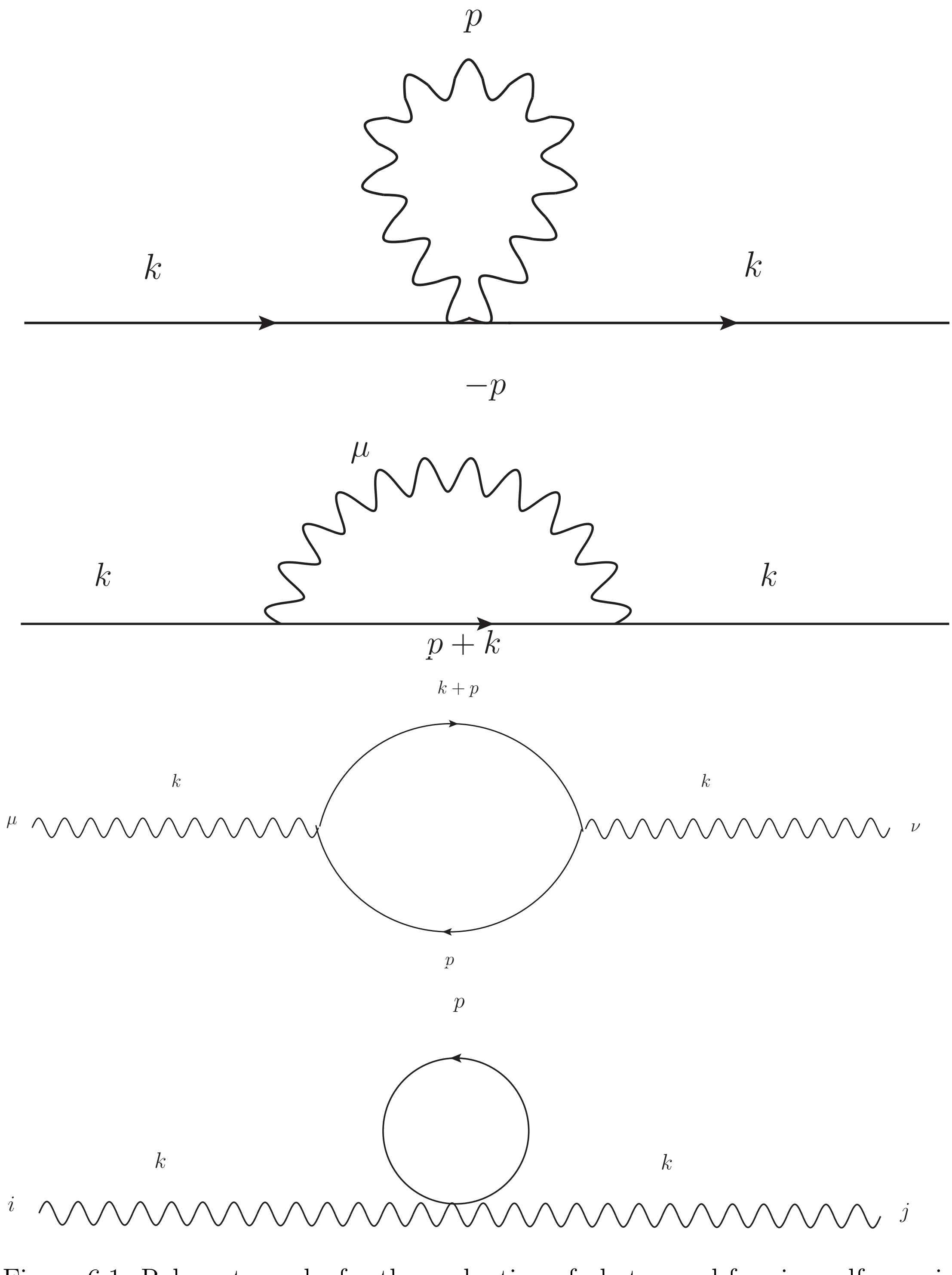

Figure 6.1: Relevant graphs for the evaluation of photon and fermion self energies.

6.3 One-loop propagators

The one-loop graphs contributing to the two-point functions are shown in fig.6.1; for both fermions and photons, the graphs involving the four-point function do not depend on the external momentum, k, and so do not contribute to the wave function renormalisation.

6.3.1 IR behaviour

In the following sections we shall take the fermion mass to be zero, as this greatly simplifies the calculations. This will unfortunately introduce IR divergences, which are regularised by dimensional continuation to $3-\epsilon$ spatial dimensions. Nevertheless, as we shall show, these divergences cancel out in the calculation of corrections to the dispersion relations.

6.3.2 UV behaviour

From naïve power counting (see eq.(6.12)), one would expect that the terms Y and Z would be divergent (logarithmically and linearly, respectively). However, dimensional regularisation leads to integrals of the form

$$\int_0^\infty \frac{q^{2a}}{(q^2+q^4)^{r-\frac{1}{2}}} dq \ , \tag{6.16}$$

(for some integers a, r), which, after dimensional continuation $2a \to 2a - \epsilon$ give, for appropriate values of ϵ

$$\frac{\Gamma\left(a - r - \frac{\epsilon}{2} + 1\right)\Gamma\left(-a + 2r + \frac{\epsilon}{2} - \frac{3}{2}\right)\big)}{2\Gamma\left(r - \frac{1}{2}\right)} \tag{6.17}$$

which can never lead to a pole of the second Gamma function in the numerator in the limit $\epsilon \to 0$, usually responsible for UV divergences. The first Gamma function may be divergent for sufficiently high r or low a, but this is an IR divergence. To check the consistency of this unusual feature, we show in the appendix B in section B.1 that corrections to the photon mass indeed vanish, as expected from gauge invariance. In the appendix C, we examine the origin of this behaviour in greater generality.

6.3.3 One-loop corrections

Details of the calculations for the self energies can be found in the Appendix, and we find

$$\begin{aligned} X &= \frac{-e^2}{8\pi^2 M}\left(\frac{1}{\epsilon}+\frac{3}{2}-\frac{A}{2}\right)+O(\epsilon) \\ Y &= \frac{-e^2}{8\pi^2 M}\left(\frac{1}{\epsilon}+\frac{7}{2}-\frac{A}{2}\right)+O(\epsilon) \\ W &= \frac{-e^2}{6\pi^2 M}\left(\frac{-1}{\epsilon}-\frac{5}{3}+\frac{A}{2}\right)+O(\epsilon) \\ Z &= \frac{-e^2}{6\pi^2 M}\left(\frac{-1}{\epsilon}+2+\frac{A}{2}\right)+O(\epsilon)\ , \end{aligned} \tag{6.18}$$

where $A=\gamma_E+2\log 2-\log\pi$ and we reiterate that the poles in ϵ correspond to IR divergences due to the massless fermion limit.
With the corrections calculated above, the IR kinetic terms of the one-loop dressed Lagrangian are

$$\begin{aligned} \mathcal{L}_{IR} &= \frac{1}{2}(1+W)F_{0i}F_{0i}-\frac{1}{4}(1+Z)F_{ij}F_{ij}M^2 \\ &\quad +\overline{\psi}(i\partial_0\gamma_0(1+X)-iM\partial_k\gamma_k(1+Y))\psi\ , \end{aligned} \tag{6.19}$$

and, in order to recover the relativistic IR propagation for photons, we rescale

$$\begin{aligned} k_0 &\to M\frac{(1+Z)^{1/4}}{(1+W)^{1/2}}\,k_0 \\ k_i &\to \frac{1}{(1+Z)^{1/4}}\,k_i \\ A_0 &\to M^{1/2}\frac{(1+Z)^{1/4}}{(1+W)^{1/2}}\,A_0 \\ A_i &\to \frac{1}{M^{1/2}(1+Z)^{1/4}}\,A_i \\ \psi &\to \frac{(1+W)^{1/4}}{(1+X)^{1/2}(1+Z)^{1/8}}\,\psi\ . \end{aligned} \tag{6.20}$$

This leads to the following effective kinetic Lagrangian

$$M^{-1}\mathcal{L}_{IR}^{eff}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+i\overline{\psi}\left[\gamma^0\partial_0-\left(1+\frac{\delta v}{2}\right)\vec{\gamma}\cdot\vec{\partial}\right]\psi\ , \tag{6.21}$$

where

$$\delta v\equiv 2Y-2X+W-Z\ , \tag{6.22}$$

and the factor M^{-1} is absorbed by the time rescaling $t\to Mt$ in the definition of the action. As expected, the photon dispersion relation is relativistic in the IR

$$k_0^2=k^2\qquad\text{(photon)}\ , \tag{6.23}$$

but the one-loop dispersion relation for fermions becomes

$$k_0^2 = m_R^2 + (1 + \delta v)k^2 \qquad \text{(fermions)} \ , \tag{6.24}$$

where m_R is the rescaled and dressed one-loop fermion mass, which we haven't taken into account up to now [1]. The product of phase and group velocities for fermions is $v_p v_g = 1 + \delta v$, and from the expressions (6.18), we can see that the $1/\epsilon$ terms cancel, leading to

$$\delta v = \frac{1}{9\pi^2}\frac{e^2}{M} \ . \tag{6.25}$$

We stress again that the rescaling laws (6.20) are UV finite, and that the poles in ϵ correspond to "artificial" IR divergences, as a consequence of the massless fermion case we study here. The expected cancellation of these artificial poles in ϵ shows the consistency of the definition of loop integrals, which should lead to an IR-divergence-free expression for δv.

6.3.4 Phenomenology

If we identify the dimensionless coupling e^2/M with $4\pi\alpha$, where $\alpha \simeq 1/137$ is the fine-structure constant, we obtain

$$\delta v \simeq 10^{-3} \ , \tag{6.26}$$

which is phenomenologically not realistic. The corresponding Lorentz-violating operators in the Standard Model Extension (SME) [8] are parametrised by the CPT-even coefficients $c_{\mu\nu}$, defined as

$$ic_{\mu\nu}\overline{\psi}\gamma^\mu\partial^\nu\psi \ . \tag{6.27}$$

In our case, we have $2c_{ij} = \delta v \ \delta_{ij}$, such that $\delta v = (2/3)\mathrm{tr}\{c_{ij}\}$. From the tables of SME coefficients [7], the later identification gives an upper bound of the order $|\delta v| \lesssim 10^{-15}$, such that one needs to consider a dimensionless coupling which satisfies

$$\frac{e^2}{4\pi M} \lesssim 10^{-14} << \alpha \ . \tag{6.28}$$

We conclude this section with a remark concerning the dressed coupling constant of the model. From the effective Lagrangian (6.21), gauge invariance ensures that the IR effective interactions are

$$-\frac{e}{M^{1/2}}\overline{\psi}\left[\gamma^0 A_0 - \left(1 + \frac{\delta v}{2}\right)\vec{\gamma}\cdot\vec{A}\right]\psi \ , \tag{6.29}$$

[1] Note that, since we did the calculations in the massless fermion case, the classical modification $1 \to (1+\eta)$ in eq.(6.5) is not present here. But, as explained below eq.(6.5), this contribution is negligible compared to the present correction due to quantum effects: $\eta << \delta v$.

such that one can define the one-loop effective coupling $e^{(1)}$ as

$$e^{(1)} \equiv e\left(1+\frac{\delta v}{2}\right) . \tag{6.30}$$

Although $e^{(1)}$ does not contain UV divergence, one can define the one-loop beta function for the evolution of the dressed coupling $e^{(1)}$ with the scale M, for fixed bare coupling e, which leads to

$$\beta \equiv M\frac{\partial e^{(1)}}{\partial M} = -\frac{1}{18\pi^2}\frac{e^3}{M} . \tag{6.31}$$

Note that this definition of beta function does not coincide with the usual one, but rather shows how the effective coupling of the model evolves as the crossover scale M between Lifshitz and relativistic regimes varies.

6.4 Conclusion

We have again shown, with a specific Lifshitz extension of QED which is classically acceptable from the phenomenological point of view, that quantum corrections change the naïve picture and lead to non-trivial Lorentz violating effects, as in the previous chapter. In order to recover a realistic theory, the model must satisfy strong constraints.

The $U(1)$ gauge symmetry forced us to introduce new derivative interactions with the UV Lifshitz scaling.

Technically speaking, it is interesting to see that, because of higher order space derivatives, the unusual powers of space momentum lead to specific regularisation features. Dimensional regularisation leads to a definition of loop integrals which makes sense even in the limit where the space dimension goes to 3. The approach is nevertheless consistent, since the physical quantities calculated here show the appropriate cancellations of UV would-be divergences (see calculation of quantum corrections for the photon mass) and cancellation of IR divergences in the case where fermions are massless (see calculation of fermion dispersion relation).

We also note that taking the limit $M \to \infty$ of the present model is not straightforward. Indeed, since the relevant (dimensionless) coupling is e^2/M, this limit cannot be taken for fixed bare parameter e without the theory becoming trivial. For this reason, the specific features of the present model do not allow one to continuously recover usual QED in the expected way as $M \to \infty$; however, one could take a simultaneous limit, holding the "physical" e^2/M constant.

Chapter 7

Hořava-Lifshitz gravity

7.1 Introduction

Hořava-Lifshitz (HL) gravity [4] is a proposed theory of modified gravity that appears to be perturbatively renormalisable, at least by power-counting. Though not without its own problems, it avoids the usual problems of UV completions of General Relativity (GR) by proposing Lifshitz scaling at high energies and exploiting the improvements to renormalisation that the inclusion of higher derivatives brings. As a Lifshitz theory in *curved and dynamical* spacetime, HL gravity is somewhat different to the previously examined models and thus merits a particular focus, which is the purpose of this chapter.

We begin by introducing HL gravity in its usual form and discuss its restricted symmetries. We then discuss the proposed extra conditions that some have applied to HL gravity in an attempt to control the large number of possible interaction terms it may possess and the various problems that even these proposals may fail to cure. In the next section, we examine the linearised classical behaviour of HL gravity, demonstrating the "extra" scalar degree of freedom. We then examine the "covariant" extension of HL gravity that will be the focus of the next chapter. Finally, we relate HL gravity to other popular deformations of GR.

For recent reviews of HL gravity, see [46] [47] [48].

Metric conventions

As this is a proposed ultraviolet completion to General Relativity, we shall adopt the mostly-plus metric convention generally used in studies of general relativity, so that what we present may be more readily compared with the existing literature. We emphasise that this is the opposite convention to that used in the preceding chapters.

7.2 Model

In HL gravity, where we shall take $z = d = 3$, it is natural to take the ADM form of the space-time metric:

$$ds^2 = -c^2 N^2 dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right) \ , \tag{7.1}$$

where c is the speed of light, with dimension $[c] = 2$. The gravitational degrees of freedom are the lapse function $N(x,t)$, the shift function $N_i(x,t)$ and the 3-dimensional space metric $g_{ij}(x,t)$. The mass dimensions of metric components are $[N] = 0$, $[N_i] = 2$, and $[g_{ij}] = 0$.

We relax the assumption of local diffeomorphism invariance

$$x_\mu \to x'_\mu(x_\nu) \tag{7.2}$$

to allow $z = 3$ Lifshitz scaling in the ultraviolet; this forces us to distinguish a time co-ordinate and space directions at each point, producing a *foliation* of the space time, the leaves of which are surfaces of constant time (and so giving a global notion of simultaneity). The symmetry group is now restricted to the *foliation-preserving* diffeomorphisms

$$x_i \to x'_i(x_j, t),\ t \to t'(t). \tag{7.3}$$

Perturbatively, these transformations can be represented by:

$$\begin{aligned}
\delta t &= f(t) \\
\delta x^i &= \xi^i(t,x) \\
\delta g_{ij} &= \partial_i \xi_j + \partial_j \xi_i + \xi^k \partial_k g_{ij} + f \dot{g}_{ij} \\
\delta N_i &= \partial_i \xi^k N_k + \xi^k \partial_k N_i + \dot{\xi}^j g_{ij} + \dot{f} N_i + f \dot{N}_i \\
\delta N &= \xi^k \partial_k N + \dot{f} N + f \dot{N} \ .
\end{aligned} \tag{7.4}$$

A generic action for HL gravity can be given by

$$S = \frac{1}{\epsilon^2} \int dt d^3x \sqrt{g} N \Big\{ K_{ij} K^{ij} - \lambda K^2 - V \Big\} \tag{7.5}$$

Where

$$K_{ij} = \frac{1}{2N}\left\{\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right\}, \quad i,j = 1,2,3 \tag{7.6}$$

is the extrinsic curvature (with $K = K_i^i$), λ a dimensionless constant, ϵ a coupling constant determining the strength of gravity (analogous to Newton's constant G in general relativity), g the determinant of the space metric.

V is an arbitrary potential made from foliation-preserving-diffeomorphism-invariant terms containing up to 6 space derivatives of the metric. There are no such invariant terms one can construct from N_i or time derivatives of N, so one need only consider the various contractions of derivatives, the 3-space Ricci tensor R_{ij} and the acceleration vector $a_i = \partial_i \ln N$ (the Riemann tensor can be expressed in terms of the Ricci tensor in 3 dimensions). It should be noted that the term $a_i a^i$ is of the same mass dimension as the R expected from General Relativity.

If the potential is allowed to contain 6-derivative terms, such as $R\Delta R$ (where R is the 3-dimensional Ricci scalar of the leaves of the foliation), then the theory is power-counting renormalisable [4]. (Indeed, as g_{ij} is dimensionless, all its couplings for interactions with up to six derivatives should be marginal or relevant). The condition $z = d$ was not arbitrary, such a choice makes the coupling ϵ (or equivalently, Newton's G) marginal.

The action for General Relativity can be recovered in the case where $\lambda = 1$ and $V = -\Lambda - c^2 R$. Note then, that there are two separate sources of Lorentz violation in this model: the higher order terms in the potential, which classically should only have noticeable effects at very high energies, and the possibilities that $\lambda \neq 1$ or that $a^i a_i$ has a non-vanishing coupling, which (classically) suffer no such suppression in the IR.

7.3 Restrictions

As there are a great number of terms that could appear in the potential, two restrictions have been proposed by Hořava:

7.3.1 Detailed Balance

Detailed balance is the suggestion that the potential should be given by

$$V = E^{ij} G_{ijkl} E^{kl} \tag{7.7}$$

where

$$\begin{aligned} G^{ijkl} &= \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - \lambda g^{ij}g^{kl} \\ \sqrt{g}E^{ij} &= \frac{\delta W[g]}{\delta g_{ij}} \end{aligned} \tag{7.8}$$

for some action W. This restriction was inspired by HL gravity's origins in condensed matter and critical phenomena, but does not seem a very sensible restriction for models in which Lorentz violation is imposed "by hand" and so shall not be considered further. Indeed, it has been found to lead to additional problems beyond those of HL gravity in general [49], such as a wrong-sign cosmological constant (for reasonable values of λ). The detailed balance condition also necessarily introduces parity violating terms to the action.

7.3.2 Projectability

Projectability (and hence "Projectable Hořava-Lifshitz gravity") is the proposal that the lapse function, N, be a function of time only. Such a condition is preserved under the foliation-preserving diffeomorphisms, as $\xi^k \partial_k N$ vanishes, and removes any terms containing spatial derivatives of $N(t)$ from the potential, leaving it a function of the 3-dimensional intrinsic curvature alone. Many important solutions in GR have a natural co-ordinate system in which this condition is true (for example: the various black hole solutions, FRWL cosmology) and it could of course always be imposed, so it does not seem too unreasonable to apply the same to HL gravity. With this condition, the number of terms in the potential becomes more manageable; if we further restrict our potential to not violate parity (as we could by including a term constructed from the Cotton tensor) there are only nine possible terms in the potential that can be expressed as

$$\begin{aligned} V_{proj} &= -\Lambda - c^2 R - \alpha_1 R^2 - \alpha_2 R_{ij} R^{ij} - \beta_1 R^3 - \beta_2 R R_{ij} R^{ij} \\ &- \beta_3 R^j_i R^k_j R^i_k - \beta_4 R \Delta R - \beta_5 \nabla_i R_{jk} \nabla^i R^{jk}, \end{aligned} \tag{7.9}$$

where all other contractions can be related to those above by some identity or an integration by parts [47]. However, the a_i terms that we have excluded here have been used to improve the IR behaviour of some Hořava gravity models [50], and can be used to reproduce MOND phenomenology (see section 7.8). Additionally, as we now have a field that depends on time only, its variation leads to a Hamiltonian constraint *integrated over space*, which can complicate Hamiltonian analyses of HL models. The extra scalar degree of freedom discussed below also appears to be unstable in projectable HL gravity, having the "wrong" sign for its low-energy kinetic terms.

7.4 Problems

7.4.1 Scalar graviton

Having removed one generator of local symmetries, it is not surprising that HL gravity contains one additional propagating degree of freedom per space-time point when compared with general relativity, normally referred to as a "scalar graviton".

We can see this by counting the degrees of freedom by the Hamiltonian method given in [51]: the number of primary constraints to take into account is the number of gauge functions *plus* the number of their time derivatives, in the situation where these gauge functions depend on both space and time. This is because gauge functions and their time derivatives must be considered independent, when defining a boundary condition for the evolution of gauge fields. In our case (not assuming projectability), we have 10 independent metric components ($N,\ N_i,\ g_{ij}$) and we see from the gauge transformations (7.4) that the functions ξ^i count twice since they appear with their time derivative, while f counts once only because it depends on t only. The total number of propagating degrees of freedom is therefore $10-(2\times3+1)=3$, consistent with the loss of one generator of symmetry.

This extra mode is problematic in that it does not generically decouple at low energies, and so it becomes difficult to recover GR in the infra-red, where the perturbative approximation to GR is very well tested. Indeed, it appears to become strongly coupled at low energies [49] at a scale around

$$M_{strong} \sim M_{pl}\sqrt{|1-\lambda|}\ , \tag{7.10}$$

or even lower [52], which is a problem when $\lambda \sim 1$ is the limit one might hope to reach in the IR and is strongly constrained by experiment.

In [52], the simultaneous limit where $M_{pl} \to \infty$ while M_{strong} (and its equivalent from $a_i a^i$) remains constant is investigated, it is found that strong coupling still generally occurs, at a lower mass scale.

Several methods have been proposed to solve this problem: [50] suggests that a non-projectable HL gravity with non-zero couplings to several of the a_i terms may remove the issue, but the couplings must be quite large.

It should be noted that this term does not appear *at* the relativistic value of $\lambda = 1$, where its kinetic term vanishes; many attempts to cure this problem try to force the theory to flow to this point in the IR (but these encounter the issues of strong coupling mentioned above). Indeed, if one considers the lowest dimension terms in the Lagrangian

$$\mathcal{L} = \sqrt{g}N[K^{ij}K_{ij} - \lambda K^2 + c^2 R + \alpha a_i a^i + \ldots] \tag{7.11}$$

and performs an expansion about flat space, one finds (after fixing the gauge and applying the linearised classical equations of motion for the non-propagating fields, see the next section for details), two distinct propagating species:

- A spin 2 graviton, as found in GR, with propagation speed c (taken to be 1 by rescaling).

- The spin 0 graviton, with propagation speed

$$c_0^2 = \frac{1-\lambda}{3\lambda-1}.\frac{\alpha-2}{\alpha} \tag{7.12}$$

from which we can obtain the condition that, for the spin 0 graviton to behave sensibly (that is $0 < c_0^2 < 1$) we require

$$\begin{gathered} 0 < \alpha < 2, \\ \lambda > 1 \text{ or } \lambda < \frac{1}{3} \end{gathered} \tag{7.13}$$

(the other sign for α is disallowed as it would give the wrong sign to a time derivative term in the action. Projectability, of course, would impose $\alpha \to \infty$, but this speed is derived from two separate terms in the expansion, one of which does not appear in the projectable case as we shall see in the next section).

It has also been argued that flowing to the fixed point at $\lambda = 1/3$ can be used to remove the unwanted scalar [53], but the region between that and the relativistic $\lambda = 1$ is disallowed by the above.

7.4.2 Quantisation and interactions

HL gravity, by naturally providing a foliation of spacetime, removes one of the difficulties in consistently quantising gravity theories, the seemingly arbitrary choice of time co-ordinate under which ones quantised theory may not be invariant. Because of this, most attempts at quantisation of this model have followed simple canonical formalisms. However, many of the problems of quantising theories of dynamical spacetime remain, and several of the methods developed for GR have been applied to HL gravity.

We note here related works, involving the quantisation of HL gravity. [54] shows that in the large N limit of projectable HL gravity, coupled to N Lifshitz scalars, the matter-induced beta functions display asymptotic freedom. Exact Renormalisation Group methods are used to provide an insight into the existence of asymptotically safe gravity [55], in that the IR relevant terms of projectable HL gravity flow to a non-gaussian fixed point in the UV (also of note in the same model is that the Einstein-Hilbert action is a saddle point with an IR-attractive direction).

Also, [56] is based on causal dynamical triangulations to study phase transitions of space time geometry in (2+1)-dimensional HL gravity. Finally, [57] describes how (1+1)-dimensional HL gravity can be quantised in a similar way as a harmonic oscillator.

There is not a unique way of coupling HL gravity to matter that may be Lifshitz-scaling itself, as one may include higher derivative terms in the matter sector also, but there do exist minimal consistent couplings for several models [58], [59].

In [60], the authors derive a general coupling to non-projectable HL gravity for scalars and vectors and then evaluate low-order propagation corrections in a manner similar to the two proceeding chapters. Interestingly, the authors find that higher derivative terms are generated dynamically even for "relativistic" matter.

7.5 Expansion about flat space

To explicitly demonstrate the classical behaviour we have discussed above, we shall now derive the linearised equations of motion for HL gravity about flat space. We shall assume projectability, as that is the most relevant case to our subsequent work.

$$\mathcal{L} = \sqrt{g}N(t)[K^{ij}K_{ij} - \lambda K^2 - V] \tag{7.14}$$

7.5.1 Constraints

Varying the action with respect to $N(t)$ yields the Hamiltonian constraint

$$\int d^3x\sqrt{g}(K^{ij}K_{ij} - \lambda K^2 + V) = 0 \tag{7.15}$$

integrated over spatial slices as N is a function of t only. Varying the shift vectors N_i gives the supermomentum constraint

$$\partial_i(K^{ij} - \lambda K g^{ij}) = 0\ . \tag{7.16}$$

7.5.2 Gauge fixing

We chose the synchronous gauge $N = 1, N_i = 0$ and expand the spatial metric about flat space as $g_{ij} = \delta_{ij} + \epsilon h_{ij}$. Our supermomentum constraint now reads (to lowest order)

$$\partial_t\partial^i[h_{ij} - \lambda h\delta_{ij}] = 0 + O(\epsilon) \tag{7.17}$$

which implies $\partial^i[h_{ij} - \lambda h\delta_{ij}] = v_j(x)$. for some vector v_i a function of space only. We still have a residual gauge transformation not fixed by the conditions above:

$$h_{ij} \to h_{ij} + \partial_i u_j(x) + \partial_j u_i(x) \tag{7.18}$$

which can be used to set $v_i = 0$ (if $\lambda \neq 1$) and so remove the vector part of h_{ij}. We can now decompose h_{ij} into a scalar part h and a traceless, transverse tensor part H_{ij} by defining the transverse tensor

$$C_{ij} = h_{ij} - \lambda h\delta_{ij} \tag{7.19}$$

and letting $H_{ij} = C_{ij} - \frac{1}{3}\delta_{ij}C_{kk}$ be the trace-free part of C_{ij}. We now have

$$h_{ij} = H_{ij} - \frac{3\lambda - 1}{2}(\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2})h + (1-\lambda)\delta_{ij}h. \tag{7.20}$$

7.5.3 Equations of motion

Now, the linearised equation of motion reads

$$\frac{1}{2}\partial_t^2[h_{ij} - (1-\lambda)\delta_{ij}h] = V_{ij} + O(\epsilon) \tag{7.21}$$

where V_{ij} is a functional derivative of the potential with respect to h_{ij}. This can be easily split into the two components above by taking a trace:

$$\begin{aligned}(1-3\lambda)\partial_t^2 h &= -2V_{kk} \\ \partial_t^2 H_{ij} &= 2V_{ij} - (\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2})V_{kk}.\end{aligned} \tag{7.22}$$

If we take $V = -R$, we obtain

$$\begin{aligned}\partial_t^2 H_{ij} &= \partial_k\partial_k H_{ij} + O(\epsilon), \\ (1-3\lambda)\partial_t^2 h &= (1-\lambda)\partial_k\partial_k h + O(\epsilon)\end{aligned} \tag{7.23}$$

almost recovering the dispersion relations from the previous section; although, as mentioned, the sign of c_0^2 is incorrect, an additional problem for the scalar graviton in the projectable case.

7.6 Cosmological bounds

A recent study [61] considered HL gravity in the context of effective field theories for modified gravity and dark energy. From this, perturbative cosmological and astrophysical predictions could be derived with standard computational packages and compared to known experimental measurements. The most strongly constrained parameters are of course those of the lowest dimension terms in the HL gravity Lagrangian, i.e.

$$\mathcal{L}_{IR} = \sqrt{g}N[K^{ij}K_{ij} - \lambda K^2 + c^2 R + \alpha a_i a^i] \tag{7.24}$$

Comparatively strict bounds are found on λ and α:

$$\begin{aligned}|1-\lambda| &< 10^{-6.2} \\ \alpha &< 10^{-2.4}\end{aligned} \tag{7.25}$$

Because of higher-order corrections we have not considered elsewhere in this chapter, c^2 is not expected to be exactly the relativistic speed of light, and so we have the additional bound $|c^2 - 1| < 0.0038$.

We additionally note that if the parameters are tuned so that the Post-Newtonian parameters used in solar system tests vanish exactly, the constraint on α becomes as stringent as that on λ.

7.7 Covariant Hořava-Lifshitz gravity

An interesting solution to the problem of the extra scalar was proposed in [62], in which an extra $U(1)$ gauge symmetry is introduced to restore a modified form of "covariance" and so bring the number of propagating degrees of freedom back to its relativistic value.

7.7.1 Symmetries and Lagrangian

We introduce an extra local symmetry by first noting that the transformation generated by $\delta N_i = \partial_i a$ for some scalar a is a symmetry of *linearised* Hořava gravity (about flat space). To close the symmetry in the full theory, we must introduce

- an auxiliary scalar field ν
- a field A that acts as a Lagrange multiplier and transforms as a vector under time reparametrisations but as a scalar under spatial transformations (i.e like N),

leading to an action of the form [1]

$$S = \frac{1}{\epsilon^2}\int dtd^3x\sqrt{g}\Big\{N\left[K_{ij}K^{ij} - \lambda K^2 - V + \nu\Theta^{ij}(2K_{ij} + \nabla_i\nabla_j\nu)\right] - A(R - 2\Omega)\Big\}, \tag{7.26}$$

where

$$\Theta^{ij} = R^{ij} - \frac{1}{2}Rg^{ij} + \Omega g^{ij} \tag{7.27}$$

and Ω is a constant. Under the new $U(1)$ symmetry the ν, A and N_i fields transform as

$$\begin{aligned}\delta_\alpha N_i &= N\nabla_i\alpha \\ \delta_\alpha A &= \dot{\alpha} - N^i\nabla_i\alpha \\ \delta_\alpha \nu &= \alpha\end{aligned} \tag{7.28}$$

and all others are invariant. Note that $[A] = 4$, $[\nu] = 1$.

7.7.2 Degree of freedom counting

By a simple extension of the degree of freedom counting argument given for normal HL gravity above, we can see that this theory will indeed possess only the two propagating degrees of freedom that we require (as we have introduced two extra constraints at the cost of one extra scalar).

[1] It has been shown in [58] that the present extension to the original HL gravity is valid for any λ, despite its original derivation at $\lambda = 1$.

Alternatively, it could be noted that the Lagrange multiplier A imposes a constant value of $R = 2\Omega$. This introduces an additional constraint to the calculations in section 7.5 (to be exact, a constant value for $\partial^2 h$) which indeed removes the scalar "trace" degree of freedom. (The extra scalar d.o.f. introduced with the auxiliary field is removed by gauge fixing.)

7.7.3 Problems

The aforementioned fixing of the value of R (i.e. the intrinsic curvature of the leaves of the preferred foliation) by the Lagrange multiplier is not problematic in a pure gravity scenario but may be considered inconsistent in the presence of matter, but this is neglecting the possibility that the "dressed" curvature may differ significantly from what we have calculated here, once all possible interactions with matter are considered (including couplings between matter fields and A).

We note that the long-distance IR limit is not obviously recovered: it has been shown in [63] that the equivalence principle is not automatically retrieved in the infra-red. The meanings of the auxiliary fields and their couplings to matter are still open questions, though the field ν is sometimes known as the "prepotential" as its divergence contributes to the Newtonian gravitational potential. It has been shown [64] that in the projectable case, one can recover seemingly relativistic IR behaviour by taking the A and ν fields to be part of the effective 4-metric for matter (in the non-projectable case, no such assumption is needed [65]).

7.7.4 Other work

This covariant extension to HL has led to several studies, including spherically symmetric solutions [66] and their relevance to an alternative model for galaxy rotation curves [67], cosmological solutions [68], as well as theoretical and phenomenological consistency tests of the theory [69].

This version of HL gravity will be discussed further in Chapter 8.

7.8 Relating HL gravity to other modified gravity models

7.8.1 Einstein-Aether and khronometric models

One of the simpler Lorentz violating modifications to General Relativity that has been considered is Einstein-Aether theory: we introduce a vector field u^μ, the "aether", restricted to be time-like and of unit length (i.e. $u^\mu u_\mu = -1$, usually imposed by a Lagrange multiplier). The most general coupling, in the IR can be given by

$$\begin{aligned} S_{EA} &= \frac{1}{16\pi G}\int d^3x dt\sqrt{g}\Big(R + \nabla_\mu u_\nu \nabla_\xi u_\rho (c_1 g^{\mu\xi}g^{\nu\rho} + c_2 g^{\mu\nu}g^{\xi\rho} + c_3 g^{\mu\rho}g^{\nu\xi} + c_4 u^\mu u^\xi g^{\nu\rho}) \\ &+ \lambda(u^\mu u_\mu + 1)\Big) \end{aligned} \tag{7.29}$$

Where the c_i are constants and λ is a Lagrange multiplier. Although this action is still "covariant" in that it is defined in terms of 4-tensors, clearly the presence of u_μ explicitly breaks Lorentz symmetry and we have a preferred local rest frame in which $u = (1, 0, 0, 0)$.

This is a very general model of low-energy Lorentz violation in the gravity sector, in order to relate it to HL gravity, we must consider a specific case: the "khronometric" model, in which u_μ is additionally assumed to be the gradient of some scalar τ (the "khronon");

$$u_\mu = \frac{\nabla_\mu \tau}{\sqrt{|\nabla_\nu \tau \nabla^\nu \tau|}} \tag{7.30}$$

suitably normalised to be of unit length. One can see that the sets of constant τ must be 3-dimensional everywhere (so that $u_\mu \neq 0$), providing a foliation of our spacetime; additionally, u_μ is unchanged by a reparametrisation $\tau \to \tau'(\tau)$.

Moving to ADM co-ordinates respecting this foliation, we can identify

$$\begin{aligned} N &= \frac{1}{\sqrt{\nabla_\nu \tau \nabla^\nu \tau}} \\ g^{(3)}_{ij} &= g_{ij} + u_i u_j \\ K_{ij} &= \nabla_i u_j \end{aligned} \tag{7.31}$$

We can then see that the interaction terms in 7.29 are reduced to those that preserve the foliation, eg.

$$\nabla_\mu u_\nu \nabla^\mu u^\nu = K^{\mu\nu}K_{\mu\nu} - a^2 \tag{7.32}$$

all of which are some combination of $R, K^2, K_{ij}K^{ij}$ and $a_i a^i$; the same relevant IR terms as are possible in HL gravity, so we conclude that khronometric and (non-projectable) Hořava models have the same low-energy classical behaviour.

7.8.2 Dark matter free models

The continued non-observation of a dark matter candidate has led some to propose modifications to gravity to explain the observed galaxy rotation curves.

MOND

MOND (MOdified Newtonian Dynamics) posits an infra-red modification to gravitational physics: a dependence on an *acceleration scale* that can be used to explain (some of) the phenomena usually attributed to dark matter. MOND acquires its name from the way it can be treated as a modification to the Poisson equation for Newtonian gravity:

$$\nabla.(f(\frac{|\nabla\phi|}{a_0})\nabla\phi) = 4\pi G\rho \tag{7.33}$$

where a_0 is the acceleration scale, and f is a function such that $f(x) \to 1$ as $x \to \infty$ and $f(x) \sim x$ as $x \to 0$. It can be shown [70], [71] that a model with a suitable function of the acceleration vector a_i in the potential can reproduce this behaviour. We can thus recover MOND at low accelerations with both khronometric and *non-projectable* HL gravity models.

Galaxy rotation curves from projectable, covariant Hořava-Lifshitz gravity

In [67] the authors attempt to emulate dark matter through an appropriate choice of the covariant HL auxiliary field A in a projectable covariant Hořava-Lifshitz model (similar to the one we shall study in the next chapter). They consider the spherically symmetric solution

$$ds^2 = -c^2N^2dt^2 + \frac{1}{f(r)}(dr + n(r)dt)^2 + r^2 d\Omega \tag{7.34}$$

where $n(r)$ is the radial component of the shift vector N_i. Taking $\lambda = 1$ (as must be approximately true), they examine the classical equations of motion, from the A and n equations, they find

$$\begin{aligned} n\partial_r f &= 0 \\ f &= 1 - \frac{B}{r} \end{aligned} \tag{7.35}$$

for some constant B. This implies either B or n vanishes; taking $n \neq 0$ and gauge fixing $\nu = 0$, the remaining equations of motion (which we shall not repeat here) lead to *multiple* solutions for n, A, given by

$$\begin{aligned} n^2 &= \frac{c}{r} - \frac{1}{2}A(r) + \frac{1}{2r}\int_0^r A(\rho)d\rho \ , \\ \int_0^\infty A'(r)dr &= 0 \end{aligned} \tag{7.36}$$

The reason one finds multiple solutions here is that the Hamiltonian constraint is integrated over space, as we are considering the projectable form of covariant HL gravity here.

The authors then attempt to recover observed galaxy rotation curves: first they set $A = 0$ outside the virial radius of the galaxy, as a boundary condition. Considering the Newton potential from these fields alone, one finds

$$\phi(r) = \frac{-n^2}{2c^2} \tag{7.37}$$

Thus, to recover any given orbital velocity profile $v(r)$ inside the galaxy, one requires

$$\frac{1}{4c^2} r A'(r) = v^2(r) + \int_0^r d\rho \frac{v^2(\rho)}{\rho} \tag{7.38}$$

which indeed has consistent solutions for physically reasonable choices of v.

It should be noted that a non-vanishing A also contributes to the vacuum energy of the universe, but this can be put within experimental bounds.

This example can be considered a clear demonstration that projectable covariant HL gravity does not necessarily recover GR at low energies, at least without a careful choice of parameters. This is discussed further in[64].

Chapter 8

Effective matter dispersion relation in quantum covariant Hořava-Lifshitz gravity

This chapter is based on a paper [3], written with Jean Alexandre.

8.1 Introduction

In this chapter, we study the effective dispersion relations of classical matter fields (a complex scalar and a photon) coupled to the covariant extension of projectable Hořava-Lifshitz gravity introduced last chapter in section 7.7. The fields are "classical" in that we treat them as external sources and integrate over the gravity degrees of freedom only, to one loop, about a flat background metric.

In this case, we take the coupling $\Omega = 0$, such that the constraint provided by the auxiliary field A consists of setting the curvature tensor to zero, $R = 0$, which we impose in the path integral over metric fluctuations. The restoring force for quantum fluctuations is then provided by higher order space derivatives of the metric, and leads to a prediction for the Lorentz-symmetry violating effective dispersion relations for these fields.

Because the matter fields are classical, the present model contains only logarithmic divergences. Additionally, our results suggest that the characteristic Hořava-Lifshitz scale should be smaller than 10^{10} GeV, if one wishes not to violate the current bounds on Lorentz symmetry violation.

The effective speed of light seen by matter interacting with HL gravity is studied in [72]. The authors derive this effective speed seen by a scalar field and an Abelian gauge field, and compare these to measure Lorentz-symmetry violation in a similar manner to the present study. However, we are treating matter as classical, and imposing the constraint $R = 0$ in the integration over graviton degrees of freedom. We emphasise that $R = 0$ is not a gauge choice, but a constraint from the additional symmetry of our model which has the physical effect of removing one degree of freedom in the theory. As described below, the constraint leads to the vanishing of one of the scalar components of the space metric.

Similarly, in [60], effective dispersion relations are calculated for "relativistic" scalars minimally coupled to conventional non-projectable HL gravity; it is found that higher-derivative terms for the scalars are dynamically generated, as might be expected generically.

A short review on the covariant version of HL gravity is presented in the next section, and the coupling to matter fields is presented in section 3, together with the integration of gravitons. This calculation is done for $\lambda \neq 1$, but the result does not depend on λ, which is a consequence of the vanishing of the trace of the fluctuating space metric. Section 4 presents a phenomenological analysis and our conclusions.

8.2 Review of covariant Hořava-Lifshitz gravity

Although it was previously described in section 7.7 we present a condensed introduction to Covariant HL gravity here.

8.2.1 Notation

Recall our convention that tensors with repeated lower indices are contracted with the flat space metric, and that

$$v^i w_i = v_i w_j g^{ij} = (g^{-1}_{ab})_{ij} v_i w_j \ . \tag{8.1}$$

Also, we denote $v^2 = v_i v_i$, we use ∂_i for the flat 3-space derivative, and ∂^2 for the flat 3-space Laplacian.

We repeat again here that we use the "mostly plus" metric signature convention in this chapter.

8.2.2 The action

We consider the $z = d = 3$, projectable, covariant HL gravity. The gravitational degrees of freedom are the lapse function $N(t)$, the shift function $N_i(x,t)$ and the 3-dimensional space metric $g_{ij}(x,t)$, which appear in the ADM form of the space-time metric 7.1. c is the speed of light, with dimension $[c] = 2$. The mass dimensions of metric components are $[N] = 0$, $[N_i] = 2$, and $[g_{ij}] = 0$.

The covariant version of HL gravity involves the auxiliary gauge fields $A(x,t)$ and $\nu(x,t)$ whose role is to impose a constraint which eliminates the scalar degree of freedom of HL gravity. The action is then given by

$$S = \frac{1}{\epsilon^2}\int dt d^3x \sqrt{g}\Big\{N\left[K_{ij}K^{ij} - \lambda K^2 - V + \nu\Theta^{ij}(2K_{ij} + \nabla_i\nabla_j\nu)\right] - AR\Big\}, \quad (8.2)$$

where

$$\begin{aligned} K_{ij} &= \frac{1}{2N}\{\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\}, \quad i,j = 1,2,3 \qquad (8.3) \\ \Theta^{ij} &= R^{ij} - \frac{1}{2}Rg^{ij} \\ V &= -c^2R - \alpha_1 R^2 - \alpha_2 R^{ij}R_{ij} - \beta_1 R^3 - \beta_2 RR^{ij}R_{ij} \\ &\quad -\beta_3 R_i^j R_j^k R_k^i - \beta_4 R\nabla^2 R - \beta_5 \nabla_i R_{jk}\nabla^i R^{jk}\ . \end{aligned}$$

Note that $[A] = 4$, $[\nu] = 1$, $[\epsilon] = 0$, the potential V includes all the renormalisable operators even under parity and we have taken the parameter $\Omega = 0$. The dimensions of the various terms in the Lagrangian are

$$[R] = 2,\ [R^2] = 4,\ [R^3] = [\Delta R^2] = 6,\ [c^2] = 4,\ [\alpha_i] = 2,\ [\beta_j] = 0\ . \quad (8.4)$$

Note that the term $R^{ijkl}R_{ijkl}$ does not appear, as the Weyl tensor in three dimensions automatically vanishes.

The action (8.2) is invariant under the following transformations:

- 3-dimensional foliation-preserving diffeomorphism

$$\begin{aligned} \delta t &= f(t) \qquad (8.5) \\ \delta x^i &= \xi^i(t,x) \\ \delta g_{ij} &= \partial_i\xi_j + \partial_j\xi_i + \xi^k\partial_k g_{ij} + f\dot{g}_{ij} \\ \delta N_i &= \partial_i\xi^k N_k + \xi^k\partial_k N_i + \dot{\xi}^j g_{ij} + \dot{f}N_i + f\dot{N}_i \\ \delta N &= \xi^k\partial_k N + \dot{f}N + f\dot{N} \\ \delta A &= \xi^k\partial_k A + \dot{f}A + f\dot{A} \end{aligned}$$

- $U(1)$ symmetry

$$\begin{aligned} \delta_\alpha N &= 0 \\ \delta_\alpha g_{ij} &= 0 \\ \delta_\alpha N_i &= N\nabla_i\alpha \\ \delta_\alpha A &= \dot{\alpha} - N^i\nabla_i\alpha \\ \delta_\alpha \nu &= \alpha \qquad (8.6) \end{aligned}$$

where α is an arbitrary spacetime function.

8.2.3 Gauge fixing and $U(1)$-symmetry constraints

The metric fluctuations h_{ij} are defined by $g_{ij} = \delta_{ij} + \epsilon h_{ij}$ and, choosing the synchronous gauge where $N = 1$ and $N_i = 0$, we decompose h_{ij} as

$$h_{ij} = H_{ij} + \partial_i V_j + \partial_j V_i + (\frac{1}{3}\delta_{ij} - \frac{\partial_i\partial_j}{\Delta})B + \frac{1}{3}\delta_{ij}h, \tag{8.7}$$

where H_{ij} is transverse traceless, V_i is transverse and h =tr$\{h_{ij}\}$.

One can easily see that the variation of the action (8.2) with respect to A leads to $R = 0$, which is a condition we will impose in the path integral over graviton degrees of freedom. This condition should be satisfied at the linear order in metric fluctuations, since we consider only quadratic terms in the gravity action. We therefore obtain

$$0 = R = -\epsilon\frac{2}{3}\partial^2(B + h) + O(\epsilon^2) \ , \tag{8.8}$$

which, together with boundary conditions $h(\infty) = B(\infty) = 0$ leads to $h = -B$ everywhere. We can re-write our expansion of h_{ij} as

$$h_{ij} = H_{ij} + \partial_i V_j + \partial_j V_i + \frac{\partial_i\partial_j}{\Delta}h. \tag{8.9}$$

Finally, We fix the $U(1)$ symmetry by setting $\nu = 0$, and we also note that ghosts decouple from matter at one loop, since the corresponding action is cubic in fluctuations of ghosts/gravitons.

8.3 Effective dispersion relation for matter fields

8.3.1 Coupling to matter

We now wish to couple the gravity sector to classical matter, including a complex scalar field ϕ and an abelian gauge field A_μ. In [58] a generic action is derived for the coupling of Covariant HL gravity to a scalar field, but we restrict ourselves to the case which recovers Lorentz symmetry in the IR

$$S_{scalar} = -\int d^3x dt \ \sqrt{g}(-\dot{\phi}\dot{\phi}^\star + c^2 g^{ij}\partial_i\phi\partial_j\phi^\star) \ , \tag{8.10}$$

where c is the speed of light, with mass dimension 2.
The coupling to an abelian gauge field A_μ is described by the action

$$S_{photon} = -\frac{1}{4}\int d^3x dt \ \sqrt{g}(-2g^{ij}F_{0i}F_{0j} + c^2 g^{ik}g^{jl}F_{ij}F_{kl}) \ , \tag{8.11}$$

and we need not worry about gauge fixing for the abelian field A_μ, since it is considered an external source. We wish to calculate effective dispersion relations for the matter fields ϕ and A_μ. As these fields couple to gravity only through their first derivatives, we may treat those derivatives as constant external fields:

- $\phi = \phi_0 \exp(ik^\mu x_\mu)$ leads to $\partial_i\phi\partial_i\phi^\star = (\vec{k})^2\phi_0^2$;
- $A_i = A_i^0 \sin(k^\mu x_\mu)$ leads to $F_{ij}^2 = 2(\vec{k})^2(A^0)^2 - 2(\vec{k}\cdot\vec{A}^0)^2 + \mathcal{O}(k^4)$.

The (anisotropic) effective action will be given by an expression of the form

$$S^{eff}_{scalar} = \int d^3xdt((1+a)\dot{\phi}\dot{\phi}^\star - (1+b)c^2\partial_i\phi\partial_i\phi^\star) \ , \tag{8.12}$$

with a corresponding dispersion relation

$$(1+a)k_0^2 = (1+b)c^2k^2 \ , \tag{8.13}$$

where we assuming $a, b \ll 1$. If we note v_ϕ^2 the product (phase velocity $\times$ group velocity), we have then

$$\frac{v_\phi^2}{c^2} - 1 = b - a + \mathcal{O}(\epsilon^3) \tag{8.14}$$

Similarly, for the photon we will obtain an effective action of the form

$$S^{eff}_{photon} = -\frac{1}{4}\int d^3xdt(-2(1+a')F_{0i}F_{0i} + c^2(1+b')F_{ij}F_{ij}) \ , \tag{8.15}$$

with the corresponding correction

$$\frac{v_A^2}{c^2} - 1 = b' - a' + \mathcal{O}(\epsilon^3) \ . \tag{8.16}$$

As noted in [72], the measurable quantity which violates Lorentz symmetry is the difference

$$\delta v^2 \equiv |v_A^2 - v_\phi^2| = c^2|b' - b - a' + a| \ . \tag{8.17}$$

8.3.2 Features of the one-loop integration

Taking into account the above gauge fixing conditions and the constraint $R = 0$ in the path integral, the gravity sector expanded to second order in the metric fluctuations reads

$$\begin{aligned} S_{gravity} \quad = \quad & -\frac{1}{4}\int d^3xdt \ H_{ij}(x)\left(\partial_t^2 + 4\alpha_2\partial^4 - 4\beta_5\partial^6\right)H_{ij} \\ & + 2\partial_iV_j\partial_t^2\partial_iV_j + (1-\lambda)h\partial_t^2h + O(\epsilon) \end{aligned} \tag{8.18}$$

Note that only the terms from the potential with no powers of R survive, and only those with two powers of R_{ij} contribute at this level, as R_{ij} is $O(\epsilon)$. We can now see that we have only two propagating degrees of freedom (the two polarisations of H_{ij}), as expected.

As the remaining two components of the metric, V_i and h, have only time derivatives, one may suspect that the theory would be unstable. This is not believed to be the case; instead, this scenario is analogous to the "conformal instability" of perturbative GR. The conformal instability can be seen if one makes a naïve decomposition of the metric perturbations into trace and trace-free parts: the kinetic term for the trace component in the Lagrangian has the wrong sign. However, careful consideration of the Jacobian factors [73] reveals that the trace term is non-propagating (which is obvious in our case) and can be re-interpreted as an auxiliary field imposing an additional constraint (but only after a *non-local* change of co-ordinates).

Indeed, in [74] similar calculations to our own are performed "on-shell" in non-projectable HL gravity (and another Lorentz-violating model) by treating the non-propagating fields as auxiliary and imposing their equations of motion as constraints in the path integral. The authors find results not dissimilar to those we shall derive shortly. A similar strategy could have been adopted here; however, the constrained action would be non-local (as the auxiliary fields appear with mixed orders of derivatives, taking their interactions with the matter fields into account) and more difficult to work with.

We may treat the cases of photons and scalars separately at the order at which we are working, as they have no interactions other than through gravity.

For the scalar we have

$$S_{scalar} = \int d^3x dt \left\{ \mathcal{L}_{\phi 0} + \epsilon^2 c^2 \left(-\frac{1}{3} H_{ij} H_{ij} - \frac{2}{3} \partial_i V_j \partial_i V_j - \frac{1}{6} h^2 \right) \partial_k \phi \partial_k \phi^\star + \mathcal{O}(\epsilon^3) \right\} \tag{8.19}$$

where $\mathcal{L}_{\phi 0}$ is proportional to $-|\dot{\phi}|^2 + c^2 \partial_i \phi \partial_i \phi^\star$, the usual relativistic Lagrangian density for the scalar. Following [72], because of isotropy in space coordinates, terms of the form $T^{ij} \partial_i \phi \partial_j \phi^\star$ that are quadratic in the graviton fields have been replaced by $(1/3) T^i_i \partial_j \phi \partial^j \phi^\star$. Terms that mix different components of the graviton cannot contribute to our corrections at $O(\epsilon^2)$ and so are neglected here. Terms linear in the graviton fields do not contribute here either, as we treat the scalars as classical external fields; for the same reason, we are able to integrate the quadratic terms by parts.

Similarly, for the photon we have

$$S_{photon} = \frac{1}{4} \int d^3x dt \left\{ \mathcal{L}_{A0} + \epsilon^2 c^2 \left(-\frac{1}{6} H_{ij} H_{ij} - \frac{1}{3} \partial_i V_j \partial_i V_j - \frac{1}{6} h^2 \right) F_{kl} F_{kl} + \mathcal{O}(\epsilon^3) \right\} , \tag{8.20}$$

where isotropy has been used and $\mathcal{L}_{A0}$ is proportional to the relativistic Lagrangian for photons.

8.3.3 Integration

As the components of the graviton do not mix, we may consider their contributions separately.

Spin 2

As we seek the *difference* between the modifications to the space and time components, we can neglect in the action the term $\mathcal{L}_{\phi 0}$. We therefore need only to consider the action

$$\tilde{S}_{scalar} = -\frac{1}{4}\int d^3x dt\ H_{ij}(x)\left(\partial_t^2 + 4\alpha_2\partial^4 - 4\beta_5\partial^6 + \frac{4\epsilon^2}{3}c^2\partial_k\phi\partial_k\phi^\star\right)H_{ij}(x)\ . \tag{8.21}$$

for the scalar and

$$\tilde{S}_{photon} = -\frac{1}{4}\int d^3x dt\ H_{ij}(x)\left(\partial_t^2 + 4\alpha_2\partial^4 - 4\beta_5\partial^6 + \frac{2\epsilon^2}{3}c^2\frac{1}{4}F_{kl}F_{kl}\right)H_{ij}(x)\ . \tag{8.22}$$

for the photon.

Scalar field

We denote by $D(q)$ the Fourier transform of $\partial_t^2 + 4\alpha_2\partial^4 - 4\beta_5\partial^6$, and we have

$$S_H = -\frac{1}{4}\int \frac{d^3p dp_0}{(2\pi)^4}\frac{d^3q dq_0}{(2\pi)^4}H_{ij}(p)\left(D(q) + \frac{4\epsilon^2}{3}c^2\partial_k\phi\partial_k\phi^\star\right)\delta(p+q)\ H_{ij}(q)\ . \tag{8.23}$$

Integrating over the two components of H, this gives a contribution to the partition function of

$$\begin{aligned}
&\left(\mathrm{Det}\left\{(D(q) + \frac{4\epsilon^2}{3}c^2\partial_k\phi\partial_k\phi^\star)\delta(p+q)\right\}\right)^{-1} \qquad (8.24)\\
= \quad & \exp\left\{-\mathrm{Tr}\left[\ln\left(D(q)\delta(p+q)\right) + \frac{4\epsilon^2}{3}c^2\partial_k\phi\partial_k\phi^\star\ D^{-1}(q)\delta(p+q)\right]\right\}\\
= \quad & \exp\left\{-\frac{4\epsilon^2}{3}c^2\partial_k\phi\partial_k\phi^\star\delta(0)\int\frac{d^3p dp_0}{(2\pi)^4}\frac{1}{-p_0^2 + 4\alpha_2 p^4 + 4\beta_5 p^6} + \cdots\right\}\ ,
\end{aligned}$$

where $\delta(0)$ is a constant global volume factor, and dots represent field-independent terms. The integral in the above leads to:

$$\begin{aligned}
I &= -\delta(0)\frac{4\epsilon^2}{3}c^2\partial_k\phi\partial_k\phi^\star\int\frac{d^3p dp_0}{(2\pi)^4}\frac{1}{-p_0^2 + 4\alpha_2 p^4 + 4\beta_5 p^6}\\
&= -\delta(0)\frac{4\epsilon^2}{3(2\pi)^4}c^2\partial_k\phi\partial_k\phi^\star \times \frac{-i\pi}{2}\int\frac{d^3p}{p^2\sqrt{\alpha_2 + \beta_5 p^2}}\ ,
\end{aligned} \tag{8.25}$$

and is logarithmically divergent. We regularise this integral by dimensional regularisation, as it respects the symmetries of the theory

$$\begin{aligned}
I_a &= -\delta(0)\frac{4\epsilon^2}{3(2\pi)^4}c^2\partial_k\phi\partial_k\phi^\star \times \frac{-i\pi}{2}\int\frac{(3-a)\pi^{\frac{3-a}{2}}p^{2-a}dp}{\Gamma(\frac{5-a}{2})p^2\sqrt{\alpha_2 + \beta_5 p^2}}\mu^a\\
&= i\delta(0)\frac{\epsilon^2}{6\pi^2}c^2\partial_k\phi\partial_k\phi^\star\frac{1}{\sqrt{\beta_5}}\frac{\mu^a}{a} + \text{finite}\ ,
\end{aligned} \tag{8.26}$$

where μ is an arbitrary mass scale. After dividing by $i\delta(0)$, in order to take into account the Wick rotation and the space time volume, the identification with the speed (8.14) gives

$$\frac{v_\phi^2}{c^2} - 1 = \frac{-\epsilon^2}{6\pi^2\sqrt{\beta_5}}\frac{\mu^a}{a} \ . \tag{8.27}$$

Photon

Comparing the coefficients of the relevant terms in the actions, one can see the effective change in velocity for the photon will be 1/2 times that of the scalar. This leads to

$$\frac{v_A^2}{c^2} - 1 = \frac{-\epsilon^2}{12\pi^2\sqrt{\beta_5}}\frac{\mu^a}{a} \ . \tag{8.28}$$

Note that both these results are sub-luminal, as might intuitively be expected: small fluctuations of the spatial metric about flat space should generally lead to a "longer" path between two points.

Spin 1

The spin 1 terms are similar to the spin 2 terms shown above, with $2\partial_i V_i \partial_i V_i$ in the place of $H_{ij}H_{ij}$. However, the calculation leads to an integral of the form

$$\int \frac{d^3p dp_0}{(2\pi)^4}\frac{1}{p_0^2} \ . \tag{8.29}$$

Unlike the previous case, the p integral here is the integral of a polynomial, which can be formally taken to vanish under dimensional regularisation, as explained in appendix C (see also [75]). The vanishing or finiteness of a regularised integral which otherwise would naïvely be divergent is explained pedagogically in [76]: in the regularised integral, divergences associated to different regions of the domain of integration cancel each other, such that the integral is finite when the regulator is removed.

Spin 0

The coefficients of the relevant terms and thus the modifications to the velocities coming from the spin 0 component of the graviton, h, are equal for the scalar and photon.[1] Hence the final result below does not depend on the parameter λ, which only appears in the spin 0 kinetic term.

[1] This is unsurprising, see the similar calculation for regular Hořava Lifshitz gravity in [72], the field the authors call σ vanishes here.

8.4 Analysis and conclusions

8.4.1 Analysis

From the results (8.27) and (8.28), the total measurable difference in the squared velocities is

$$\delta v^2 \equiv v_\phi^2 - v_A^2 = \frac{-\epsilon^2 c^2}{12\pi^2\sqrt{\beta_5}} \frac{\mu^a}{a} + \text{finite} . \tag{8.30}$$

We define the beta function, in the limit $a \to 0$, by

$$\beta = \mu \frac{\partial(\delta v^2)}{\partial \mu} = \frac{-\epsilon^2 c^2}{12\pi^2\sqrt{\beta_5}} , \tag{8.31}$$

and we can then write, for some mass scale μ_0,

$$\delta v^2 = \frac{-\epsilon^2 c^2}{12\pi^2\sqrt{\beta_5}} \ln \frac{\mu}{\mu_0} . \tag{8.32}$$

In order to select a reasonable value for μ_0, we repeat the calculation of the integral (8.25), regularised by the high momentum cut-off Λ to obtain

$$\begin{aligned} I_\Lambda &= i\delta(0)\frac{\epsilon^2}{6\pi^2} c^2 \partial_k\phi\partial_k\phi^\star \frac{1}{\sqrt{\beta_5}} \ln\left(\sqrt{\frac{\beta_5}{\alpha_2}}\Lambda + \sqrt{1 + \frac{\beta_5}{\alpha_2}\Lambda^2}\right) \\ &= i\delta(0)\frac{\epsilon^2}{12\pi^2} c^2 \partial_k\phi\partial_k\phi^\star \frac{1}{\sqrt{\beta_5}} \ln \frac{\Lambda^2}{\alpha_2} + \text{finite} , \end{aligned} \tag{8.33}$$

such that the identification with the speed (8.14) gives

$$\frac{v_\phi^2}{c^2} - 1 = \frac{-\epsilon^2}{12\pi^2\sqrt{\beta_5}} \ln \frac{\Lambda^2}{\alpha_2} , \tag{8.34}$$

and suggests $\mu_0 = \sqrt{\alpha_2}$ as a natural choice.

The result (8.32) is obtained in anisotropic Minkowski space time, and we rescale the time coordinate as $t \to t\ M_{HL}^2$, where M_{HL} is a scale characteristic of HL gravity, below which the classical model can be considered relativistic. Speeds are then rescaled as $v \to v\ M_{HL}^{-2}$, and we set the speed of light in isotropic Minkowski space time to 1 (we therefore identify the dimensionful quantity $c = M_{HL}^2$). The coupling constant appearing in the action (8.2) is then $\epsilon = M_{HL}/M_{Pl}$, where M_{Pl} is the Planck mass, and the measurable deviation from Lorentz symmetry in the effective theory is

$$\delta v^2 = |v_\phi^2 - v_A^2| = \frac{M_{HL}^2}{24\pi^2 M_{Pl}^2\sqrt{\beta_5}} \ln \frac{\mu^2}{\alpha_2} , \tag{8.35}$$

which should be less than about 10^{-20} [7].

8.4.2 Conclusions

In order to get an idea of the order of magnitude for M_{HL}, one can set the different parameters to natural values in the present context, which are $\beta_5 \simeq 1$,

$\alpha_2 \simeq M_{HL}^2$ and $\mu \simeq M_{Pl}$. This shows that one should take $M_{HL} \lesssim 10^{10}$ GeV for the result (8.35) to satisfy the upper bounds on Lorentz-symmetry violation. This value is also obtained in [77], where non-relativistic corrections to matter kinetic terms are calculated in the framework of another 4-dimensional diffeomorphism breaking gravity model, and where 10^{10} GeV corresponds to the cut off above which the model is no longer valid. We note that this scale also corresponds to the Higgs potential instability [78], which could be avoided by taking into account quantum gravity effects in the calculation of the Higgs potential [79], and it would be interesting to look for a stabilising mechanism in the framework of non-relativistic gravity models.

We comment here on the relevance of Lorentz-symmetry violation in the study of cosmic rays, with energies necessarily lower than the Greisen-Zatsepin-Kuzmin cutoff $E_{GZK} = 10^{19.61 \pm 0.03}$eV [80], since the latter is of the order of the bound we find for M_{HL}. Above this energy, protons interact with the Cosmic Microwave Background, producing pions which decay and generate showers observable on Earth. As noted in [81], resulting photon-induced showers would be highly sensitive to Lorentz-symmetry violation, and the observation of 10^{19}eV photons would put strong bounds on the different Lorentz-symmetry violating parameters of the Standard Model Extension (SME) [8]. The observation of such photons would also help put bounds on parameters in the expression (8.35), which could then be related to the SME. The bound $\delta v^2 \lesssim 10^{-25}$ found in [81] from potential ultra-high-energy photons, is actually much smaller than the one considered above, and concerns electron/positrons created by these high energy photons in the presence of the Earth magnetic field. Taking into account this bound, and assuming β_5 of order 1, we find the typical upper bound $M_{HL} \lesssim 10^6$ GeV. Although this is still well above the current accessible energies at CERN, the observation of $\sim 10^{19}$ eV photons would definitely put much stronger constraints on Lorentz-symmetry violation.

We also remark that supersymmetric models have been studied in the context of Lifshitz-type theories [82], where an interesting feature is that particles in the same supermultiplet see the same limiting speed. Furthermore, in the case of supersymmetric gauge theory, the limiting speed is the same for matter and gauge supermultiplets. From the phenomenological point of view, it is suggested that the Lorentz-symmetry violation scale M_{HL} should be above the SUSY breaking scale, in order to avoid fine-tuning problems imposed by bounds on Lorentz-symmetry violation. This means that M_{HL} should be at least of the order of 10 TeV, which is well below the bound we find here.

We make a final comment regarding the calculation carried out in [72], for the original, non-covariant Hořava-Lifshitz gravity and where a *quadratic* divergence is found. This stems not from the extra degree of freedom, but from treating the matter as quantum fields and then considering terms quartic in the matter fields, obtained from completing the square for the coupling terms linear in the graviton fields, that we neglected above. In terms of Feynman diagrams, treating matter as quantum fields consists in considering self energy graphs with an internal matter propagator, which would lead to integrals that go as

$$\int dk_0 d^3k \frac{k^2}{k_0^2(k_0^2 - k^2)} \tag{8.36}$$

from the h and V_i components, which are indeed quadratically divergent [2]. Our analysis shows that, as long as matter is classical, only logarithmic divergences arise.

[2] of course, such divergences are invisible to dimensional regularisation and should not affect the perturbative β functions

Chapter 9

Conclusions

The main observation of this thesis is that interacting Lifshitz-scaling theories can have much more significant Lorentz-violating effects at low energies than would be expected from a purely classical analysis, once low-order quantum corrections are taken into account. In this chapter, we shall summarise the preceding work, make a few general observations on the common features of all the models studied and present some ideas for future work.

9.1 Summary

In this thesis, we have investigated Lifshitz-type quantum field theories in several different contexts, using effective actions to derive low-energy phenomenological consequences to the classical high-energy Lorentz symmetry violation found in such theories; in particular, the modification of infra-red dispersion relations which are readily observable. We began by introducing the basic features of Lorentz violating models in chapter 2 and the methods we would need to compute the relevant quantum corrections in the context of effective actions in chapter 3. In chapter 4 we introduced the concept of Lifshitz-scaling theories themselves, and discussed their effects on renormalisability and dimensional analysis.

Chapters 5 and 6 contained original research, in which the concepts discussed in the previous sections were applied to two very different theories of interacting fermions.

In the first, investigated in chapter 5, the fermions were considered as models for neutrinos; several of the useful properties of Lifshitz theories were demonstrated, such as the renormalisability of interactions that would be non-renormalisable in an isotropic theory and the dynamical generation of masses (and, by the nature of the interaction, flavour oscillations). This latter effect allowed bounds to be placed on the coupling constants of the theory. Also demonstrated was the main topic of this thesis, that, although the theory was classically Lorentz invariant at low energies, quantum corrections produced overly large anisotropies in the dispersion relations. In this case, the corrections were too significant for the model to be treated as physical in its current context.

In the second, in chapter 6, a more complicated model was investigated: fermions in an abelian gauge theory. In this chapter, we encountered several of the difficulties involved with combining Lifshitz-scaling with gauge invariance: maintaining both may necessitate the introduction of new interactions not seen in the relativistic case, and the "natural" choices of gauge fixing term are significantly altered. Changing the effective number of dimensions also allowed us to demonstrate some of the peculiar behaviour of the method of dimensional regularisation (see appendix C for details). Finally, it was again found that the effective dispersion relations can differ significantly from what one would expect classically.

In chapter 7, we introduced Hořava-Lifshitz gravity, and discussed its various forms and problems. In particular, we discussed the "covariant" extension of the theory, which would be of use in the next chapter. We also demonstrated that Hořava-Lifshitz gravity, in the infra-red, behaves like a certain limit of Einstein-Aether gravity and can reproduce the effects of MOND and similar theories.

Finally, in chapter 8, we presented another piece of original research in which the techniques developed in the previous sections were applied to the case of spin-2 particles, which originally motivated the study of Lifshitz theories in high-energy physics. We considered the effect on matter fields of integrating out quantum fluctuations of the metric in the covariant extension of Hořava-Lifshitz gravity. Once again, we found significant effects on the low-energy propagation relations of the matter fields, which allowed us, by comparison to experimental limits, to place sensible bounds on the characteristic scale of Lorentz violation in such theories.

9.2 Analysis

9.2.1 Dressed dispersion relations

We have repeatedly found that quantum corrections in Lifshitz QFTs "drag down" Lorentz violating effects from high energies into the infra-red. This is perhaps unsurprising: consider the typical propagator for a Lifshitz-scaling particle:

$$D(p_\mu) = \frac{1}{p_0^2 - M^{2z-2}p^2 \pm p^{2z}} \tag{9.1}$$

In a loop integral, this propagator may appear with an argument similar to $p_\mu = r_\mu + k_\mu$, where r is an internal loop momentum, and k an external momentum. If we were to look for the terms relevant to the dressed dispersion relations here, we would expand the above to second order in k_μ. In such an expansion, there would only be two terms proportional to k_0^2, being $\frac{-1}{(r_0^2 - M^{2z-2}r^2 \pm r^{2z})^2}$ and $\frac{4r_0^2}{(r_0^2 - M^{2z-2}r^2 \pm r^{2z})^3}$, whereas the terms proportional to $k^2 = k_i k_i$ will appear with some sum of powers of binomial coefficients from expanding $(r+k)^{2z}$. Additionally, as k_0 has a greater mass dimension than k, we would in general expect the k_0^2 and k^2 terms to diverge differently under the integral over r.

We should therefore consider it quite unusual if two fields, coupled differently to a Lifshitz-scaling sector, did *not* exhibit different low-energy dispersion relations, without some symmetry protecting them.

Throughout this thesis, we have worked at low orders, one or two loops at most; one might wonder if there is some higher-order effect that we are missing. However, as we have demonstrated, it is the higher-order terms in the loop expansion whose divergence Lifshitz theories reduce; all the models discussed herein are (power-counting) renormalisable, and so we should not expect any higher-order terms to be more divergent than those we considered. Even if such terms were comparable, they are suppressed by powers of the couplings. This, of course, neglects the possibility of non-perturbative effects, such as those we exploited to calculate dynamical masses in chapter 5.

9.2.2 Mass scales

As mentioned at the end of chapter 8, the limit $M \leq 10^{10}$ GeV found for the mass scale of Covariant Hořava-Lifshitz gravity is suggestive, as it is similar to scales found elsewhere in high-energy physics (in particular the Higgs instability). Of what, precisely, it is suggestive remains to be seen, but any Lifshitz-scaling sector is likely to induce knock-on effects in all other sectors to which it is coupled and so we suggest that the Lifshitz crossover scale (which may vary between species) is a sensible place to expect "new physics", in any model that possess such a scale.

Indeed, there is no reason for the anisotropies in the effective actions we consider to stop at second order in derivatives. It seems one Lifshitz-scaling field may induce similar behaviour in other fields to which it couples, such as is found in [60]. It is interesting to note, however, that these induced higher-derivative terms would appear proportional to some power of the couplings, as well as the Lifshitz scale M, so different fields could, a priori, exhibit *different* typical mass scales for Lorentz violation.

9.3 Further Work

It is clear that more work needs to be done on determining exactly when a relativistic field theory can be considered as a limiting case of a Lifshitz-scaling one. The difficulties in producing a sensible low-energy limit of Hořava gravity are well-documented, but even our simple QED model had an ambiguous meaning to the limit $M \to \infty$. Establishing under what circumstances such limits can be taken may allow one to find new behaviours in Lorentz-invariant theories that are more easily seen in the Lifshitz case but survive under the limit.

Gauge invariance, which tends to be imposed on Lorentz-violating extensions of existing theories, is originally derived from considerations of unitarity under Poincaré transformations of massless spin 1 states. It should perhaps be more carefully examined under what circumstances this reasoning may extend to Lorentz-violating theories, either as an exact or approximate symmetry.

Studies of Lifshitz-scaling theories tend to be focused on individual models that are then taken as proofs-of-concept for the properties they display. Perhaps the large, and possibly observable departures from relativistic propagation we derived in this work can be demonstrated for broader categories of model.

In the other direction, it may be of interest to investigate how Lifshitz scaling may be induced in relativistic theories, or those with more "normal" forms of Lorentz violation, such as are introduced by some schemes of gravity quantisation. Conversely, the production of the lower-order kinetic terms in a Lifshitz theory, such as is found in [17], could be considered a spontaneous breaking of the anisotropic re-scaling symmetry, and approached from that angle.

The induction of Lifshitz-like behaviour by one field in another, as in [60] may be an interesting topic to study: firstly, it might provide a natural separation of scales. Secondly, the effect should be mutual: it may be possible to construct some self-consistency relation for the scale of Lorentz violation, perhaps finding an isolated "non-perturbative" solution for $M < \infty$.

Inflation is highly sensitive to ultra-violet effects (for instance, one generically expects a large renormalisation of the slow-roll parameters); while the effects of modified gravity theories, including Hořava-Lifshitz, on inflationary signals have been studied, it is mostly in the context of effective field theory. There may be some more subtle effects of UV Lifshitz scaling that such studies would not see. For instance, the derivation of the Lyth bound relies on the dimensionality of the inflaton during large field excursions.

Non-perturbative phenomena should generally be altered by the introduction of Lifshitz scaling. Not particular to Lifshitz models, one could imagine that in a theory with broken Lorentz symmetry there may be vacuum states that “would” be related by, say, a boost transformation that are rendered inequivalent by the breaking of that symmetry; there might then be instanton solutions tunnelling between these two states.

While much has been said about the ability of Lifshitz scaling to render normally nonrenormalisable interactions marginal, the same process can reduce a renormalisable model to a *super*-renormalisable one (such as the QED model in chapter 6). Super-renormalisable models have very restricted classes of non-perturbative effects (all-orders resummation effects may still be present, but otherwise the perturbation series is exact); this may lead to the Lifshitz-scaling model behaving very differently from its relativistic equivalents, but does not seem to have been studied in much depth.

Appendix A

Two-loop propagator in the 4-fermi model

An expansion in the frequency k_0 of the integrand appearing on the right-hand side of eq.(5.50) gives

$$\frac{N(-p_0,-\vec{p})N(-q_0,-\vec{q})N(p_0+q_0+k_0,\vec{p}+\vec{q})}{D(-p_0,-\vec{p})D(-q_0,-\vec{q})D(p_0+q_0+k_0,\vec{p}+\vec{q})} = \frac{N(-p)N(-q)N(p+q)}{D(-p)D(-q)D(p+q)} \tag{A.1}$$
$$+ k_0\gamma^0\frac{N(-p)N(-q)}{D(-p)D(-q)D(p+q)} - 2k_0(p_0+q_0)\frac{N(-p)N(-q)N(p+q)}{D(-p)D(-q)D^2(p+q)} + \mathcal{O}(k_0^2)\ ,$$

where $(p)\equiv(p_0,\vec{p})$, and an expansion in the spatial momentum $\vec{k}$ gives

$$\frac{N(-p_0,-\vec{p})N(-q_0,-\vec{q})N(p_0+q_0,\vec{p}+\vec{q}+\vec{k})}{D(-p_0,-\vec{p})D(-q_0,-\vec{q})D(p_0+q_0,\vec{p}+\vec{q}+\vec{k})} = \frac{N(-p)N(-q)N(p+q)}{D(-p)D(-q)D(p+q)} \tag{A.2}$$
$$- \frac{N(-p)N(-q)}{D(-p)D(-q)D(p+q)}\left[2((\vec{p}+\vec{q})\cdot\vec{k})((\vec{p}+\vec{q})\cdot\vec{\gamma}) + (M^2+(\vec{p}+\vec{q})^2)(\vec{k}\cdot\vec{\gamma})\right]$$
$$+ \frac{N(-p)N(-q)N(p+q)}{D(-p)D(-q)D^2(p+q)}2((\vec{p}+\vec{q})\cdot\vec{k})\left[M^2+(\vec{p}+\vec{q})^2\right]\left[M^2+3(\vec{p}+\vec{q})^2\right] + \mathcal{O}(k_0^2)\ .$$

The first term in the k_0-expansion leads to the integral

$$(k_0\gamma^0)\int_{p,q}\frac{N(-p)N(-q)}{D(-p)D(-q)D(p+q)} \tag{A.3}$$
$$= (k_0\gamma^0)\int_{p,q}\frac{p_0q_0-(M^2+\vec{p}^2)(M^2+\vec{q}^2)\vec{p}\cdot\vec{q}}{D(-p)D(-q)D(p+q)}$$
$$+(k_0\gamma^0)\int_{p,q}\frac{p_0(M^2+\vec{q}^2)\vec{q}\cdot\vec{\gamma}-q_0(M^2+\vec{q}^2)\vec{p}\cdot\vec{\gamma}}{D(-p)D(-q)D(p+q)}\ ,$$

and, because of the symmetry $p\leftrightarrow q$, the second integral vanishes. The following rescaling

$$p_0 = M^3u_0\ ,\quad q_0 = M^3v_0\ ,\quad \vec{p} = M\vec{u}\ ,\quad \vec{q} = M\vec{v}\ , \tag{A.4}$$

together with a Wick rotation on u_0, v_0 finally leads to

$$
\begin{aligned}
& (k_0\gamma^0)\int_{p,q}\frac{N(-p)N(-q)}{D(-p)D(-q)D(p+q)} && \text{(A.5)}\\
= & \; -(k_0\gamma^0)\int\frac{d^4u_E}{(2\pi)^4}\frac{d^4v_E}{(2\pi)^4}\;\frac{u_4v_4+(1+\vec{u}^2)(1+\vec{v}^2)\vec{u}\cdot\vec{v}}{D_E(u_E)D_E(v_E)D_E(u_E+v_E)}\;, && \text{(A.6)}
\end{aligned}
$$

where $D_E(u_E) = u_4^2 + (1+\vec{u}^2)^2\vec{u}^2$.

The second term in the k_0-expansion gives

$$
\begin{aligned}
& -2k_0\int_{u,v}(p_0+q_0)\frac{N(-p)N(-q)N(p+q)}{D(-p)D(-q)D^2(p+q)} && \text{(A.7)}\\
= & \; -2(k_0\gamma^0)\int_{p,q}(p_0+q_0)^2\frac{p_0q_0-(M^2+\vec{p}^2)(M^2+\vec{q}^2)\vec{p}\cdot\vec{q}}{D(-p)D(-q)D^2(p+q)}\\
& -2k_0\int_{p,q}(p_0+q_0)\frac{p_0\gamma^0\;\vec{q}\cdot\vec{\gamma}\;(\vec{p}+\vec{q})\cdot\vec{\gamma}\;(M^2+q^2)[M^2+(\vec{p}+\vec{q})^2]-\;(p\leftrightarrow q)}{D(-p)D(-q)D^2(p+q)}\;,
\end{aligned}
$$

where, by symmetry, the terms proportional to $\vec{\gamma}$ lead to a vanishing integral. After the rescaling (A.4) and a Wick rotation, we then obtain

$$
\begin{aligned}
& -2k_0\int_{u,v}(p_0+q_0)\frac{N(-p)N(-q)N(p+q)}{D(-p)D(-q)D^2(p+q)} && \text{(A.8)}\\
= & \; 2(k_0\gamma^0)\int\frac{d^4u_E}{(2\pi)^4}\frac{d^4v_E}{(2\pi)^4}\;(u_4+v_4)^2\;\frac{u_4v_4+(1+\vec{u}^2)(1+\vec{v}^2)\vec{u}\cdot\vec{v}}{D_E(u_E)D_E(v_E)D_E^2(u_E+v_E)}\;.
\end{aligned}
$$

The term proportional to $k_0\gamma^0$ is then

$$
(k_0\gamma^0)\int\frac{d^4u_E}{(2\pi)^4}\frac{d^4v_E}{(2\pi)^4}\;\frac{u_4v_4+(1+\vec{u}^2)(1+\vec{v}^2)\vec{u}\cdot\vec{v}}{D_E(u_E)D_E(v_E)D_E(u_E+v_E)}\left(-1+\frac{2(u_4+v_4)^2}{D_E(u_E+v_E)}\right)\;. \quad \text{(A.9)}
$$

For the first term in the $\vec{k}$-expansion, we use the identity

$$
\int_{p,q}f(p,q)\;\vec{p}\cdot\vec{k}\;\;(\vec{p}+\vec{q})\cdot\vec{\gamma} = \frac{\vec{k}\cdot\vec{\gamma}}{3}\int_{p,q}f(p,q)\;\vec{p}\cdot(\vec{p}+\vec{q})\;, \quad \text{(A.10)}
$$

where $f(p,q)$ depends on $(\vec{p})^2, (\vec{q})^2$ and $\vec{p}\cdot\vec{q}$ only. The rescaling (A.4) and a Wick rotation then lead to the integral

$$
\begin{aligned}
& -\int_{p,q}\frac{N(-p)N(-q)}{D(-p)D(-q)D(p+q)}\left[2((\vec{p}+\vec{q})\cdot\vec{k})((\vec{p}+\vec{q})\cdot\vec{\gamma})+(M^2+(\vec{p}+\vec{q})^2)(\vec{k}\cdot\vec{\gamma})\right]\\
= & \; M^2(\vec{k}\cdot\vec{\gamma})\int\frac{d^4u_E}{(2\pi)^4}\frac{d^4v_E}{(2\pi)^4}\left(1+\frac{5}{3}(\vec{u}+\vec{v})^2\right)\frac{u_4v_4+(1+\vec{u}^2)(1+\vec{v}^2)\vec{u}\cdot\vec{v}}{D_E(u_E)D_E(v_E)D_E(u_E+v_E)}\;. \quad \text{(A.11)}
\end{aligned}
$$

The second term in the $\vec{k}$-expansion leads to the integral

$$
\begin{aligned}
& 2\int_{p,q}\frac{N(-p)N(-q)N(p+q)}{D(-p)D(-q)D^2(p+q)}((\vec{p}+\vec{q})\cdot\vec{k})\left[M^2+(\vec{p}+\vec{q})^2\right]\left[M^2+3(\vec{p}+\vec{q})^2\right]\\
= & \; -2M^2\int_{u,v}\frac{u_0v_0-(1+\vec{u}^2)(1+\vec{v}^2)\vec{u}\cdot\vec{v}}{D_E(u_E)D_E(v_E)D_E^2(u_E+v_E)}(1+(\vec{u}+\vec{v})^2)(\vec{u}+\vec{v})\cdot\vec{\gamma}\\
& \times(\vec{u}+\vec{v})\cdot\vec{k}\;\;[1+(\vec{u}+\vec{v})^2][1+3(\vec{u}+\vec{v})^2]\;, && \text{(A.12)}
\end{aligned}
$$

where, by symmetry, the term not proportional to $\vec{\gamma}$ vanishes. Using the identity (A.10), a Wick rotation then leads to

$$\begin{aligned}
& 2\int_{p,q} \frac{N(-p)N(-q)N(p+q)}{D(-p)D(-q)D^2(p+q)}((\vec{p}+\vec{q})\cdot\vec{k})\left[M^2+(\vec{p}+\vec{q})^2\right]\left[M^2+3(\vec{p}+\vec{q})^2\right] \\
= & -\frac{2}{3}M^2(\vec{k}\cdot\vec{\gamma})\int \frac{d^4u_E}{(2\pi)^4}\frac{d^4v_E}{(2\pi)^4}\frac{u_4v_4+(1+\vec{u}^2)(1+\vec{v}^2)\vec{u}\cdot\vec{v}}{D_E(u_E)D_E(v_E)D_E^2(u_E+v_E)} \\
& \times(\vec{u}+\vec{v})^2[1+(\vec{u}+\vec{v})^2]^2[1+3(\vec{u}+\vec{v})^2] \ . \qquad \text{(A.13)}
\end{aligned}$$

The term proportional to $(\vec{k}\cdot\vec{\gamma})$ is then

$$\begin{aligned}
& M^2(\vec{k}\cdot\vec{\gamma})\int \frac{d^4u_E}{(2\pi)^4}\frac{d^4v_E}{(2\pi)^4}\frac{u_4v_4+(1+\vec{u}^2)(1+\vec{v}^2)\vec{u}\cdot\vec{v}}{D_E(u_E)D_E(v_E)D_E(u_E+v_E)} \\
& \times\left(1+\frac{5}{3}(\vec{u}+\vec{v})^2-\frac{2}{3}(\vec{u}+\vec{v})^2\frac{[1+(\vec{u}+\vec{v})^2]^2[1+3(\vec{u}+\vec{v})^2]}{D_E(u_E+v_E)}\right) \qquad \text{(A.14)}
\end{aligned}$$

Finally, from eqs.(A.9,A.14), the quantum corrections to the IR dispersion relation are determined by

$$Y_a - Z_a = 4g_a^2(3g_a^2+4g_b^2)\int \frac{d^4u_E}{(2\pi)^4}\frac{d^4v_E}{(2\pi)^4}\ \mathrm{Int}\ , \qquad \text{(A.15)}$$

where the integrand is

$$\begin{aligned}
\mathrm{Int} = & \ \frac{1}{3}\frac{u_4v_4+(1+\vec{u}^2)(1+\vec{v}^2)\vec{u}\cdot\vec{v}}{D_E(u_E)D_E(v_E)D_E^2(u_E+v_E)} \qquad \text{(A.16)} \\
& \times\left[(u_4+v_4)^2\left(6+5(\vec{u}+\vec{v})^2\right)-(\vec{u}+\vec{v})^2(1+(\vec{u}+\vec{v})^2)^2(2+(\vec{u}+\vec{v})^2)\right] \ .
\end{aligned}$$

Note that in the Lorentz-symmetric case, higher orders in $\vec{u},\vec{v}$ are absent and $Y_a = Z_a$. The integral (A.15) is evaluated as follows.

We can first perform the exact integration over u_4, v_4, using the Feynman parametrisation. This introduces two new variables of integration, but which lie in a compact domain of integration:

$$\begin{aligned}
& \frac{1}{D_E(u_E)D_E(v_E)D_E^2(u_E+v_E)} \\
= & \ 6\int_0^1 dx\int_0^{1-x} dy\frac{1-x-y}{[xD_E(u_E)+yD_E(v_E)+(1-x-y)D_E(u_E+v_E)]^4} \ ,
\end{aligned}$$

We then introduce the variables a, b, such that

$$u_4 = s(a+b) \quad \text{and} \quad v_4 = t(a-b)\ , \quad \text{with} \quad s=\sqrt{1-x} \quad \text{and} \quad t=\sqrt{1-y}\ , \qquad \text{(A.17)}$$

to obtain

$$\frac{du_4\ dv_4}{D_E(u_E)D_E(v_E)D_E^2(u_E+v_E)} = \int_0^1 dx\int_0^{1-x} dy\frac{12\ da\ db\ st\sigma}{[2st(st+\sigma)a^2+2st(st-\sigma)b^2+D]^4} \ ,$$

where

$$\begin{aligned} D &= x(1+u^2)^2u^2 + y(1+v^2)^2v^2 + \sigma(1+\Sigma)^2\Sigma \\ \Sigma &= (\vec{u}+\vec{v})^2 \quad , \qquad \sigma = 1-x-y \; . \end{aligned} \tag{A.18}$$

We then write, with $0 \leq \rho < \infty, 0 \leq \phi < 2\pi$

$$\sqrt{2st(st+\sigma)}\; a = \rho\cos\phi \qquad \text{and} \qquad \sqrt{2st(st-\sigma)}\; b = \rho\sin\phi \tag{A.19}$$

to obtain

$$\int du_4 \int dv_4 \;\mathrm{Int} \tag{A.20}$$

$$\begin{aligned} &= 2\int_0^1 dx \int_0^{1-x} dy \frac{\sigma}{\sqrt{s^2t^2-\sigma^2}} \int_0^\infty \rho d\rho \int_0^{2\pi} d\phi \frac{1}{[\rho^2+D]^4} \\ &\quad \times \left[\frac{\rho^2}{2}\left(\frac{\cos^2\phi}{st+\sigma} - \frac{\sin^2\phi}{st-\sigma}\right) + (1+u^2)(1+v^2)\vec{u}\cdot\vec{v}\right] \\ &\quad \times \left[\frac{\rho^2}{2st}\left(\frac{(s+t)^2\cos^2\phi}{st+\sigma} + \frac{(s-t)^2\sin^2\phi}{st-\sigma}\right)(6+5\Sigma) - \Sigma(1+\Sigma)^2(2+\Sigma)\right] \\ &= 2\pi \int_0^1 dx \int_0^{1-x} dy \frac{\sigma}{\sqrt{s^2t^2-\sigma^2}} \int_0^\infty \rho d\rho \frac{A\rho^4 + B\rho^2 + C}{[\rho^2+D]^4} \\ &= \frac{\pi}{6}\int_0^1 dx \int_0^{1-x} dy \frac{\sigma}{\sqrt{s^2t^2-\sigma^2}} \left(\frac{2A}{D} + \frac{B}{D^2} + \frac{2C}{D^3}\right) \; , \end{aligned} \tag{A.21}$$

where

$$\begin{aligned} A &= \frac{6+5\Sigma}{4} \; \frac{2s^2t^2 + 4\sigma^2 - 3\sigma(s^2+t^2)}{(s^2t^2-\sigma^2)^2} \\ B &= (1+u^2)(1+v^2)\vec{u}\cdot\vec{v}(6+5\Sigma)\frac{s^2+t^2-2\sigma}{s^2t^2-\sigma^2} + \Sigma(1+\Sigma)^2(2+\Sigma)\frac{\sigma}{s^2t^2-\sigma^2} \\ C &= -2(1+u^2)(1+v^2)\vec{u}\cdot\vec{v}\;\Sigma(1+\Sigma)^2(2+\Sigma) \; . \end{aligned} \tag{A.22}$$

We then define $\vec{u}\cdot\vec{v} = uv\cos\theta$ and

$$u = r\cos\alpha \quad , \quad v = r\sin\alpha \quad , \quad \text{with} \quad 0 \leq r < \infty \;\; \text{and} \;\; 0 \leq \alpha \leq \pi/2 \; , \tag{A.23}$$

and the final integral is

$$F(\Lambda/M) = \int \frac{d^4u_E}{(2\pi)^4}\frac{d^4v_E}{(2\pi)^4} \;\mathrm{Int} \tag{A.24}$$

$$= \frac{1}{3\times 2^8\pi^5}\int_0^1 dx \int_0^{1-x} dy \int_0^{\Lambda/M} dr \int_0^\pi d\theta \int_0^{\pi/2} d\alpha \; \frac{\sigma\; r^5\sin^2(2\alpha)\sin\theta}{\sqrt{s^2t^2-\sigma^2}} \left(\frac{2A}{D} + \frac{B}{D^2} + \frac{2C}{D^3}\right) \; ,$$

which is quadratically divergent, as $F(z) \sim \kappa z^2$ when $z \to \infty$. We then find via numerical integration

$$\kappa = \lim_{z\to\infty}\left\{\frac{1}{2z}\frac{dF}{dz}\right\} \tag{A.25}$$

$$\begin{aligned} &= \lim_{r\to\infty}\left\{\frac{r^4}{3\times 2^9\pi^5}\int_0^1 dx \int_0^{1-x} dy \int_0^\pi d\theta \int_0^{\pi/2} d\alpha \; \frac{\sigma\;\sin^2(2\alpha)\sin\theta}{\sqrt{s^2t^2-\sigma^2}}\left(\frac{2A}{D} + \frac{B}{D^2} + \frac{2C}{D^3}\right)\right\} \\ &\simeq -3.49\times 10^{-5} \; , \quad (\text{to a } 1\% \text{ accuracy}) \; . \end{aligned} \tag{A.26}$$

Appendix B

Details of loop integrals in $z = 2$ QED

Throughout this chapter, we shall use dimensionless momentum 4-vectors. In the following, $k_\mu = (M^2 r_0, M\vec{r})$ is the external momentum, and $p_\mu = (M^2 q_0, M\vec{q})$ is a loop momentum to be integrated over (for scalar q_0 and 3-vector q). The graphs are calculated by first integrating over frequencies and then using the three-dimensional spherical co-ordinates, with external space momentum $\vec{r} = (0, 0, r)$ and loop momentum $\vec{q} = (q\cos\theta\sin\phi, q\sin\theta\sin\phi, q\cos\phi)$. In all but two relevant cases (those used to calculate Z and the space part of the fermion integral) the integrands are a function of only $|q|, q_0$ and the angle ϕ between $\vec{r}$ and $\vec{q}$. The integration measure for q in $3-\epsilon$ dimensions can therefore be written as:

$$\int d^{3-\epsilon}q = \int_0^\infty q^{2-\epsilon}dq \int_0^\pi \sin^{1-\epsilon}\phi d\phi \int d\Omega_{2-\epsilon} \tag{B.1}$$

Where $d\Omega_{2-\epsilon}$ represents the integral over the remaining angular variables, which evaluates to

$$2\frac{\pi^{\frac{2-\epsilon}{2}}}{\Gamma(\frac{2-\epsilon}{2})} \tag{B.2}$$

In the few cases where the remaining angles cannot be eliminated, the index structure shows that they will appear only at quadratic order, as q_1^2 or q_2^2. We can thus safely make the substitution $q_1^2 \to q^2 \frac{1}{2-\epsilon}\sin^2\phi$ wherever such terms appear.

In what follows, we note

$$\int_q \equiv \int \frac{d^{3-\epsilon}q dq_0}{(2\pi)^{4-\epsilon}} \ . \tag{B.3}$$

B.1 Photon mass

As a check, we shall ensure the photon mass correction vanishes, as implied by the Ward identity. For $\mathcal{D}_{00}$ and $\mathcal{D}_{0i}$ we need consider only the vacuum polarisation graph, but for $\mathcal{D}_{ij}$ we must consider both photon graphs shown in the two lower

figures of fig. 1.
The $(i,0)$ term is proportional to the integral

$$\mathrm{Tr}\int_q \frac{(\gamma_i+2q_i)(q_0\gamma_0-q_k\gamma_k+q^2)\gamma_0(q_0\gamma_0-q_k\gamma_k+q^2)}{(q_0^2-q^2-q^4)^2}\ . \tag{B.4}$$

Every term in the numerator is either traceless or proportional to an odd power of q and thus vanishes.
The $(0,0)$ term is

$$\mathrm{Tr}\int_q \frac{\gamma_0(q_0\gamma_0-q_k\gamma_k+q^2)\gamma_0(q_0\gamma_0-q_k\gamma_k+q^2)}{(q_0^2-q^2-q^4)^2} = \int_q \frac{4(q_0^2+q^2+q^4)}{(q_0^2-q^2-q^4)^2} = 0\ . \tag{B.5}$$

This vanishes due to the q_0 integration and so is unaffected by dimensional regularisation.
Finally, the (i,j) term from the vacuum polarisation is

$$\mathrm{Tr}\int_q \frac{(\gamma_i+2q_i)(q_0\gamma_0-q_k\gamma_k+q^2)(\gamma_j+2q_j)(q_0\gamma_0-q_k\gamma_k+q^2)}{(q_0^2-q^2-q^4)^2}\ , \tag{B.6}$$

which, with dimensional regularisation, gives a finite value of $-4/3\pi^2$ in the limit $\epsilon\to 0$ (which we can take as there is no IR divergence). Similarly, the graph with a four-point vertex gives

$$2\mathrm{Tr}\int_q \frac{(q_0\gamma_0-q_k\gamma_k+q^2)}{(q_0^2-q^2-q^4)}\ , \tag{B.7}$$

which evaluates to $4/3\pi^2$, cancelling the other contribution.

B.2 Fermion corrections

Writing $e\Gamma_\mu$ for the vertex functions, with $\Gamma_0=\gamma_0, \Gamma_i = M\gamma_i + 2p_i^\psi + p_i^A$, we are looking for the terms linear in r_0 and r in the expression

$$\begin{aligned} & e^2M^{1-\epsilon}\int_q \Gamma^\mu\ G(q+r)\ \Gamma^\nu D_{\mu\nu}(q) \qquad\qquad (\text{B.8}) \\ = \ & e^2M^{1-\epsilon}\int_q\Bigg\{-\frac{\gamma_0[\gamma_0(q_0+r_0)-\gamma_l(q_l+r_l)+(q+r)^2]\gamma_0(q^2+1)}{(q_0^2-q^2-q^4)((q_0+r_0)^2-(q+r)^2(q+r)^4)} \\ + \ & \frac{(\gamma_i+2r_i+q_i)[\gamma_0(q_0+r_0)-\gamma_l(q_l+r_l)+(q+r)^2](\gamma_i+2r_i+q_i)}{(q_0^2-q^2-q^4)((q_0+r_0)^2-(q+r)^2(q+r)^4)}\Bigg\}\ , \end{aligned}$$

where G and D are the fermion and photon propagators, respectively. Using the notation of eq.(6.15), we find

$$\begin{aligned} Y &= \frac{-e^2}{M}\Bigg((3-\epsilon)\frac{2^{\epsilon-7}\pi^{-2+\frac{\epsilon}{2}}(3+\epsilon-\epsilon^2)\Gamma(\frac{-\epsilon}{2})\Gamma(\frac{1+\epsilon}{2})}{\Gamma(\frac{5-\epsilon}{2})} \\ &\qquad - \frac{2^{\epsilon-6}\pi^{-2+\frac{\epsilon}{2}}\Gamma(\frac{-\epsilon}{2})\Gamma(\frac{1+\epsilon}{2})}{\Gamma(\frac{3-\epsilon}{2})}\Bigg) \\ &= \frac{-e^2}{8M\pi^2}\left(\frac{1}{\epsilon}+\frac{7}{2}-\frac{A}{2}\right)+O(\epsilon)\ , \end{aligned} \tag{B.9}$$

where $A = \gamma_E + 2\log 2 - \log \pi$. Similarly, we find

$$X = \frac{-e^2}{8\pi^2 M}\left(\frac{1}{\epsilon} + \frac{3}{2} - \frac{A}{2}\right) + O(\epsilon) \ , \tag{B.10}$$

so that the contribution to the velocity correction is

$$2(Y - X) = \frac{-e^2}{2\pi^2 M} \ . \tag{B.11}$$

B.3 Photon corrections

We shall calculate W from the term quadratic in external momentum of $\mathcal{D}_{00}$, hence we need the terms proportional to r^2 from the integral

$$e^2 M^{1-\epsilon} \ \mathrm{Tr} \int_q \frac{\gamma_0(\gamma_0(q_0 + r_0) - \gamma_l(q_l + r_l) + (q+r)^2)\gamma_0(q_0\gamma_0 - q_k\gamma_k + q^2)}{(q_0^2 - q^2 - q^4)((q_0 + r_0)^2 - (q+r)^2 - (q+r)^4)} \ , \tag{B.12}$$

and we find

$$\begin{aligned} W &= \frac{-e^2}{M}\frac{2^{\epsilon-3}\pi^{-2+\frac{\epsilon}{2}}(2\epsilon - 3)\Gamma(\frac{-\epsilon}{2})\Gamma(\frac{3+\epsilon}{2})}{3\Gamma(\frac{5-\epsilon}{2})} \\ &= \frac{-e^2}{6\pi^2 M}\left(\frac{-1}{\epsilon} - \frac{5}{3} + \frac{A}{2}\right) + O(\epsilon) \ . \end{aligned} \tag{B.13}$$

To calculate the term Z, we need to evaluate $\mathcal{D}_{ij}$ at one loop. Having done so in $d = 3$, we obtain a matrix Π_{ij} of integrals, being the terms proportional to r^2 in

$$\begin{aligned} e^2 M^{3-\epsilon} \ \mathrm{Tr} \int_q &\frac{(\gamma_i + 2q_i + r_i)(\gamma_0(q_0 + r_0) - \gamma_l(q_l + r_l) + (q+r)^2)}{(q_0^2 - q^2 - q^4)((q_0 + r_0)^2 - (q+r)^2 - (q+r)^4)} \\ &\times (\gamma_j + 2q_j + r_j)(q_0\gamma_0 - q_k\gamma_k + q^2) \ . \end{aligned} \tag{B.14}$$

As expected, given the configuration of the external space momentum, the off-diagonal terms and Π_{33} vanish upon integration. We have

$$\Pi_{11} = \Pi_{22} = -Z \tag{B.15}$$

we find

$$\begin{aligned} Z &= \frac{e^2}{M}\frac{2^{\epsilon-5}\pi^{\frac{\epsilon-3}{2}-\frac{1}{2}}(\epsilon - 5)(\epsilon(\epsilon + 10) - 3)\Gamma\left(-\frac{\epsilon}{2}\right)\Gamma\left(\frac{\epsilon+1}{2}\right)}{3\Gamma\left(\frac{7}{2} - \frac{\epsilon}{2}\right)} \\ &= \frac{-e^2}{6\pi^2 M}\left(\frac{-1}{\epsilon} + 2 + \frac{A}{2}\right) + O(\epsilon) \ . \end{aligned} \tag{B.16}$$

The corresponding correction is then

$$W - Z = \frac{11e^2}{18\pi^2 M} \ . \tag{B.17}$$

Appendix C

Dimensional Regularisation

The method of dimensional regularisation (that is, analytically continuing the number of spacetime dimensions, d, away from an integer value) is frequently used as a method of regularisation in gauge theories, as it allows one to preserve the Ward identity (or its equivalents) in the regularised theory, unlike a high-momentum cuttoff which may violate that symmetry. "Divergences" are then represented as poles in the parameter $\epsilon = d - d_0$, where d_0 is the usual number of dimensions (4 in most cases).

As we work with Lorentz-violating theories in which there is a distinguished time co-ordinate, we shall in practice analytically continue the number of *space* dimensions only.

C.1 Finiteness of naïvely divergent integrals

There is a feature of dimensional regularisation, not often seen in the integrals derived from four dimensional QFT, but fairly common in odd dimensions (and the anisotropic models with which we work in this thesis); that is, an integral that appears UV divergent in d_0 dimensions may have an expression in $d \neq d_0$ that can be analytically continued back to d_0 without encountering any poles.

This is shown explicitly in chapter 6 where an integral, for $\epsilon \neq 0$, can be written as a product of Gamma functions that can be analytically continued in the usual fashion back to $\epsilon = 0$ to give a finite answer. It can also be seen in the following toy example: consider the following integral of a form one often sees in QFT calculations (where n is usually the number of dimensions, plus a constant)

$$I(n) = \int_0^\infty dx \frac{x^n}{(k^2 + m^2)^3} \tag{C.1}$$

If one were to impose a UV cutoff, one would see that this integral diverges as Λ^{n-5} and so one would certainly expect it to be divergent at, say, $n = 6$. However, away from integer values of n, this integral can be evaluated exactly as

$$I(n) = \frac{m^{n-5}}{(5-n)} \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{7-n}{2})}{\Gamma(3)} \tag{C.2}$$

This expression can then be continued analytically to the whole plane, less a few isolated singularities; in particular, it is regular at $n = 6$ (and any other even value of n). Such behaviour is generic in analytic continuation, indeed, it is a general property of limits and one should not be overly surprised.

In general, dimensional regularisation can "see" only *logarithmic* divergences (that is, those that would go as $\log \Lambda$ if we had introduced a cutoff), whereas those with purely polynomial divergences give finite contributions as $\epsilon \to 0$. This can be seen schematically as follows: consider a polynomially divergent integral in d dimensions, $I(d)$, we shall isolate its divergent part as

$$I(d) = (finite) + C \int d^d k k^n \ , \tag{C.3}$$

where C is some constant coefficient. Simple dimensional analysis gives $[\int d^d k k^n] = n + d$ and the second term above would indeed diverge as Λ^{n+d}, were there a cutoff. However, the integral above has *no mass scales* and so, if it is to have any finite value at all, it must vanish. Such a result is of course, easily analytically continued to almost all values of d. So for consistency, under dimensional regularisation, we must take

$$\int d^d k k^n = 0, \quad (d + n \neq 0) \ . \tag{C.4}$$

The only exception to the above is at $d + n = 0$, where the integral is dimensionless and the divergence is logarithmic. One could consider this the $m \to 0$ limit of equation C.2.

This behaviour is not a problem for renormalisation group studies with dimensional regularisation, as it is the logarithmically divergent terms that determine the beta functions. The higher-order divergences are of course still "encoded" in the regularised expression, as poles at other integer values of ϵ.

C.2 Divergence cancelling

We emphasise that the cases considered above are technically distinct from the finite results obtained for integrals such as

$$\int d^d x \frac{4x_\mu x_\nu - x^2 \eta_{\mu\nu}}{(x^2 - m^2)^3} \tag{C.5}$$

at $d = 4$, as are discussed in [76]. By power counting this integral is logarithmically divergent and indeed is not well-defined at $d = 4$. If one splits the integrand into several parts, or divides the domain of integration into several angular regions, one finds terms in each that require regularisation and give logarithmic divergences. However, upon summing these terms, the divergences cancel exactly, leaving a finite result as $d \to 4$.

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